Online Appendix for "Ambiguity, Information Processing, and Financial Intermediation"

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Online Appendix A: Comparative Statics Analysis

The risk premium. Given that

$$\pi_{R,t}^{U}=\frac{\sigma^2(1+\theta_1/\rho)\left(1+\theta_1^h/\rho^h\right)}{\rho^2 x_t\left[1+(\rho^h-\rho)x_t\right]}\frac{\left[\left(\rho^h-\rho\right)+\left(\theta_1^h-\theta_1\right)+(1+\theta_1/\rho)\frac{1}{x_t}\right]}{\left[\left(1+\theta_1^h/\rho^h\right)-(1+\theta_1/\rho)+(1+\theta_1/\rho)\frac{1}{\rho x_t}\right]^2},$$

we have

$$\frac{d\pi_{R,t}^{U}}{d\theta_1}=\frac{2\sigma^2\left(\gamma^h\right)^2\left[\rho^h+\theta_1^h+\gamma\left(1/x_t-\rho\right)\right]}{\rho^3 x_t\left[1+(\rho^h-\rho)x_t\right]\left[\gamma^h-\gamma+\gamma/\left(\rho x_t\right)\right]^3}.$$

To evaluate the sign of this expression, we consider the following possibilities. Possibility (1): If $\gamma^h>\gamma\geq 1$, we have

$$x_t>0>-\frac{\gamma}{\rho\left(\gamma^h-\gamma\right)}\Leftrightarrow\gamma^h-\gamma+\gamma\frac{1}{\rho x_t}>0.$$

Possibility (2): If $\gamma>\gamma^h\geq 1$,

$$\frac{\gamma}{\gamma-\gamma^h}>1\Leftrightarrow\frac{\gamma}{\rho\left(\gamma-\gamma^h\right)}>\frac{1}{\rho}\geq x_t\Leftrightarrow\gamma^h-\gamma+\gamma\frac{1}{\rho x_t}>0.$$

Possibility (3): If $\gamma^h=\gamma\geq 1$,

$$\gamma^h-\gamma+\gamma\frac{1}{\rho x_t}=\gamma\frac{1}{\rho x_t}>0.$$

We thus have:

$$\frac{d\pi_{R,t}^{U}}{d\theta_1}=\frac{2\sigma^2\left(\gamma^h\right)^2\left[\rho^h+\theta_1^h+\gamma\left(1/x_t-\rho\right)\right]}{\rho^3 x_t\left[1+\varDelta\rho x_t\right]\left[\gamma^h-\gamma+\gamma/\left(\rho x_t\right)\right]^3}>0.$$

Given that

$$\pi_{R,t}^{U}=\frac{1}{P/D}\frac{\sigma^{2}\left(1+\theta_{1}/\rho\right)\beta_{t}^{U}}{x_{t}\left[\rho^{h}-\left(\rho^{h}-\rho\right)\beta_{t}^{U}\right]^{2}},$$

we have

$$\begin{aligned}\frac{d\pi_{R,t}^{U}}{d\theta_{1}^{h}}&=\frac{1}{P/D}\frac{\sigma^{2}\left(1+\theta_{1}/\rho\right)}{x_{t}\left[\rho^{h}-\left(\rho^{h}-\rho\right)\beta_{t}^{U}\right]^{3}}\left[\frac{d\beta_{t}^{U}}{d\theta_{1}^{h}}\left(\rho^{h}-\left(\rho^{h}-\rho\right)\beta_{t}^{U}\right)+2\left(\rho^{h}-\rho\right)\frac{d\beta_{t}^{U}}{d\theta_{1}^{h}}\beta_{t}^{U}\right]\\&=\frac{\rho^{h}}{\left[1+\left(\rho^{h}-\rho\right)x_{t}\right]x_{t}}\frac{\sigma^{2}\left(1+\theta_{1}/\rho\right)^{2}\left(1/x_{t}-\rho\right)\left(\rho^{h}+\left(\rho^{h}-\rho\right)\beta_{t}^{U}\right)\left(\beta_{t}^{U}\right)^{2}}{\left(\rho^{h}-\left(\rho^{h}-\rho\right)\beta_{t}^{U}\right)^{3}\rho^{h}\left(1+\theta_{1}^{h}/\rho^{h}\right)}.\end{aligned}$$

Since $0\leq\beta_{t}^{U}\leq1$, $\rho\leq\rho^{h}-\left(\rho^{h}-\rho\right)\beta_{t}^{U}\leq\rho^{h}$ and

$$\frac{d\pi_{R,t}^{U}}{d\theta_{1}^{h}}=\frac{\sigma^{2}\gamma^{2}\left(1/x_{t}-\rho\right)\left(\rho^{h}+\Delta\rho\beta_{t}^{U}\right)\left(\beta_{t}^{U}\right)^{2}}{\gamma^{h}x_{t}\left(1+\Delta\rho x_{t}\right)\left(\rho^{h}-\Delta\rho\beta_{t}^{U}\right)^{3}}\geq0.$$

The Sharpe ratio. Given the expression for $\pi_{R,t}^{U}/\sigma_{R,t}^{U}$, we have

$$\begin{aligned}\frac{d\left(\pi_{R,t}^{U}/\sigma_{R,t}^{U}\right)}{d\theta_{1}}&=\frac{\sigma\left(\rho^{h}+\theta_{1}^{h}\right)^{2}x_{t}}{\left[\left(\rho\theta_{1}^{h}-\rho^{h}\theta_{1}\right)x_{t}+\rho^{h}\left(1+\theta_{1}/\rho\right)\right]^{2}}=\frac{\sigma\gamma^{h2}x_{t}}{\left[\rho\left(\gamma^{h}-\gamma\right)x_{t}+\gamma\right]^{2}}>0,\\\frac{d\left(\pi_{R,t}^{U}/\sigma_{R,t}^{U}\right)}{d\theta_{1}^{h}}&=\frac{\sigma\left(1+\theta_{1}/\rho\right)\rho^{h}\left(\theta_{1}+\rho\right)\left(\frac{1}{\rho}-x_{t}\right)}{\left[\left(\rho\theta_{1}^{h}-\rho^{h}\theta_{1}\right)x_{t}+\rho^{h}\left(1+\theta_{1}/\rho\right)\right]^{2}}\frac{\sigma\gamma^{2}\left(1-\rho x_{t}\right)}{\rho^{h}\left[\rho\left(\gamma^{h}-\gamma\right)x_{t}+\gamma\right]^{2}}\geq0,\\\frac{d\left(\pi_{R,t}^{U}/\sigma_{R,t}^{U}\right)}{d\bar{\theta}_{1}}&=\frac{\sigma\left(1+\bar{\theta}_{1}/\rho\right)\left[\Delta\rho\left(\rho^{h}-\frac{\bar{\theta}_{1}^{2}}{\rho}\right)x_{t}+\rho^{h}\left(1+\bar{\theta}_{1}/\rho\right)^{2}\right]}{\left[-\Delta\rho\bar{\theta}_{1}x_{t}+\rho^{h}\left(1+\bar{\theta}_{1}/\rho\right)\right]^{2}}.\end{aligned}$$

When $\Delta\rho\left(\rho^{h}-\frac{\bar{\theta}_{1}^{2}}{\rho}\right)x_{t}+\rho^{h}\left(1+\bar{\theta}_{1}/\rho\right)^{2}>0$, $\frac{d\left(\pi_{R,t}^{U}/\sigma_{R,t}^{U}\right)}{d\bar{\theta}_{1}}>0$ and

$$\left\{\begin{array}{ll}\rho^{h}-\frac{\bar{\theta}_{1}^{2}}{\rho}=0 & \Leftrightarrow\bar{\theta}_{1}=\sqrt{\rho\rho^{h}}\\\left\{\begin{array}{ll}\rho^{h}-\frac{\bar{\theta}_{1}^{2}}{\rho}<0 & \Leftrightarrow\bar{\theta}_{1}>\sqrt{\rho\rho^{h}}\\x_{t}<-\frac{\rho^{h}\left(1+\bar{\theta}_{1}/\rho\right)^{2}}{\Delta\rho\left(\rho^{h}-\frac{\bar{\theta}_{1}^{2}}{\rho}\right)} & =\phi\end{array}\right.\end{array}\right.$$

Taking first derivative of ϕ with respect to $\bar{\theta}_{1}$ yields:

$$\frac{d\phi}{d\bar{\theta}_{1}}=\frac{-2\rho^{h}\left(1+\bar{\theta}_{1}/\rho\right)}{\Delta\rho\left(\rho\rho^{h}-\bar{\theta}_{1}^{2}\right)^{2}}\left(\rho^{h}-\bar{\theta}_{1}-2\frac{\bar{\theta}_{1}^{2}}{\rho}\right).$$

Setting it to be zero gives:

$$\bar{\theta_1} = \frac{-\rho \pm \sqrt{\rho^2 + 8\rho\rho^h}}{4}.$$

Since $\bar{\theta_1} \geq 0$, we only consider the positive root. We now show $\bar{\theta_1} = \frac{-\rho+\sqrt{\rho^2+8\rho\rho^h}}{4} < \sqrt{\rho\rho^h}$. Note that $\frac{\bar{\theta_1}}{\sqrt{\rho\rho^h}} = \frac{-\sqrt{\frac{\rho}{\rho^h}}+\sqrt{\frac{\rho}{\rho^h}+8}}{4} < \frac{3}{4} < 1$, where we use the fact that $0 < \frac{\rho}{\rho^h} \leq 1$. Then we have $\rho^h - \bar{\theta_1} - 2\frac{\bar{\theta_1}^2}{\rho} < 0$ for $\bar{\theta_1}\sqrt{\rho\rho^h}$, which implies that $\frac{d\phi}{d\bar{\theta_1}} > 0$.

Now we show that $\phi > 1/\rho$. Given that $-\frac{\rho^h\left(1+\bar{\theta_1}/\rho\right)^2}{\Delta\rho\left(\rho^h-\frac{\bar{\theta_1}^2}{\rho}\right)} > \frac{1}{\rho}$, we have

$$-\rho\rho^h\left(1+\frac{2\bar{\theta_1}}{\rho}+\frac{\bar{\theta_1}^2}{\rho}\right) > \left(\rho^h\right)^2 - \frac{\rho^h}{\rho}\bar{\theta_1}^2 - \rho\rho^h + \bar{\theta_1}^2 \text{ or } \left(\bar{\theta_1}+\rho^h\right)^2 > 0,$$

which means that $x_t < \frac{1}{\rho} < \phi$. We then have $\frac{d\left(\pi^U_{R,t}/\sigma^U_{R,t}\right)}{d\bar{\theta_1}} > 0$.

The Exposure price. Given that $\frac{dk_t}{d\bar{\theta_1}} = \frac{\sigma^2(1+m)}{\left(m\rho^h+\rho\right)^2}\frac{\rho^h(1-\rho x_t-\rho m x_t)}{\left[1+(\rho^h-\rho)x_t\right]x_t}$ and

$$\bar{x}^c = \frac{1}{\tilde{m}\rho^h+\rho} = \frac{1}{\rho}\frac{1}{1+\frac{\rho^h+\bar{\theta_1}}{\rho+\bar{\theta_1}}m} \leq \frac{1}{\rho}\frac{1}{1+m},$$

we have

$$x_t \leq \bar{x}^c \leq \frac{1}{\rho}\left(\frac{1}{1+m}\right) \text{ or } 1-\rho x_t\left(1+m\right) > 0, \tag{1}$$

which means that

$$\frac{dk_t}{d\bar{\theta_1}} = \frac{\sigma^2(1+m)}{\left(m\rho^h+\rho\right)^2}\frac{\rho^h\left[1-\rho x_t\left(1+m\right)\right]}{\left[1+(\rho^h-\rho)x_t\right]x_t} > 0.$$

The interest rate. Denote the Sharpe ratio $sp \equiv \frac{\pi_{R,t}}{\sigma_{R,t}}$, the expression for the interest rate can be written as:

$$r_t = \rho^h + \hat{g}_t - \rho\left(\rho^h-\rho\right)x_t - \rho q_t x_t - \frac{\rho x_t\left[\left(sp/\gamma\right)^2 - 2\sigma\left(sp/\gamma\right)\right]+\sigma^2}{1-\rho x_t}.$$

Taking derivative with respect to θ_1 gives:

$$\frac{dr_t}{d\theta_1} = -\rho x_t\frac{dq_t}{d\theta_1} - \frac{2x_t}{\left(1-\rho x_t\right)\gamma^2}\left[\left(\frac{sp}{\gamma}-\sigma\right)\left(-sp+\frac{dsp}{d\theta_1}\rho\gamma\right)\right], \tag{2}$$

which means that in the unconstrained case,

$$\frac{dr_t^U}{d\theta_1} = \frac{2\sigma^2 x_t \gamma^h (1-\rho x_t)}{\left[\rho\left(\gamma^h-\gamma\right)x_t+\gamma\right]^3}\left(\gamma^h-\gamma\right) = \begin{cases} \frac{dr_t^U}{d\theta_1} < 0, & \text{if } \gamma^h < \gamma \\ \frac{dr_t^U}{d\theta_1} \geq 0, & \text{if } \gamma^h \geq \gamma. \end{cases}$$

Similarly, we have

$$\frac{dr_t}{d\theta_1^h} = -\rho x_t \frac{dq_t}{d\theta_1^h} - \frac{2\rho x_t}{(1-\rho x_t)\gamma}\left(\frac{sp}{\gamma}\frac{dsp}{d\theta_1^h} - \sigma\frac{dsp}{d\theta_1^h}\right), \tag{3}$$

which means that

$$\frac{dr_t^U}{d\theta_1^h} = -\frac{2\sigma^2\rho\gamma x_t(1-\rho x_t)\left(\gamma^h-\gamma\right)}{\rho^h\left[\rho\left(\gamma^h-\gamma\right)x_t+\gamma\right]^3}\begin{cases} \frac{dr_t^U}{d\theta_1^h} > 0, & \text{if } \gamma^h < \gamma \\ \frac{dr_t^U}{d\theta_1^h} \leq 0, & \text{if } \gamma^h \geq \gamma. \end{cases}.$$

In addition,

$$\begin{aligned}\frac{dr_t^U}{d\bar{\theta}_1} &= \frac{2\sigma^2\bar{\theta}_1 \Delta\rho x_t\left[\Delta\rho\left(\rho^h\bar{\theta}_1+\rho\rho^h\right)x_t - \Delta\rho\rho^h\left(1+\bar{\theta}_1/\rho\right)\right]}{\rho\left(1+\bar{\theta}_1/\rho\right)\left[-\Delta\rho\bar{\theta}_1 x_t+\rho^h\left(1+\bar{\theta}_1/\rho\right)\right]^3} \\ &= -\frac{2\sigma^2\bar{\theta}_1\rho^h(\Delta\rho)^2 x_t(1-\rho x_t)}{\rho\left[-\Delta\rho\bar{\theta}_1 x_t+\rho^h\left(1+\bar{\theta}_1/\rho\right)\right]^3} \leq 0.\end{aligned}$$

In the constrained case, from Equations (2) and (3), we have

$$\begin{aligned}\frac{dr_t}{d\theta_1} &= -\frac{\sigma^2 m}{\left(m\rho^h+\rho\right)^2 x_t} - \frac{2x_t}{(1-\rho x_t)\gamma^2}\left[\left(\frac{\sigma}{\left(m\rho^h+\rho\right)x_t}-\sigma\right)\left(-\frac{\sigma\gamma}{\left(m\rho^h+\rho\right)x_t}+\frac{\sigma(\rho+\theta_1)}{\rho\left(m\rho^h+\rho\right)x_t}\right)\right] \\ &= -\frac{\sigma^2 m}{\left(m\rho^h+\rho\right)^2 x_t} < 0, \\ \frac{dr_t}{d\theta_1^h} &= \frac{\rho\sigma^2 m^2}{\left(m\rho^h+\rho\right)^2(1-\rho x_t)} > 0, \\ \frac{dr_t}{d\bar{\theta}_1} &= -\frac{\sigma^2 m\left[1-\rho x_t(1+m)\right]}{\left(m\rho^h+\rho\right)^2(1-\rho x_t)x_t} < 0\end{aligned}$$

where we used the condition, (1).

Online Appendix B: Stationary Specialist Wealth Distribution

Without filtering, the stationary specialist wealth distribution can be solved explicitly. Specifically, let $G = 1 - \frac{\gamma^h}{\gamma} \in [0,1]$ and $\Xi = \frac{\sigma \frac{\gamma^h}{\gamma}}{1-G\rho x}$, we have

$$\begin{aligned}\mu_u(x) &= \sigma^2 + \varDelta\rho(1-\rho x) - \frac{\rho x\left(\Xi^2 - 2\sigma\Xi\right) + \sigma^2}{1-\rho x} + \Xi^2 - \sigma\Xi \\ &= \sigma^2 + \varDelta\rho(1-\rho x) + \sigma^2 \frac{\left(\gamma^h/\gamma\right)^2(1-2\rho x) + \left(\gamma^h/\gamma\right)(1-\rho G x)(3\rho x - 1) - (1-\rho G x)^2}{(1-\rho x)(1-\rho G x)^2}\end{aligned}$$

and

$$\sigma_u^2(x) = \sigma^2 G^2 \left(\frac{1-\rho x}{1-\rho G x}\right)^2.$$

We can then compute that:

$$\begin{aligned}\int \frac{2\mu_u(x)}{\sigma_u^2(x)} dx = \frac{2}{G^2}\Bigg[&\int \left(\frac{1-\rho G s}{1-\rho s}\right)^2 ds + \frac{\varDelta\rho}{\sigma^2}\int \frac{(1-\rho G s)^2}{1-\rho s} ds + \left(\frac{\gamma^h}{\gamma}\right)^2 \int \frac{1-2\rho s}{(1-\rho s)^3} ds \\ &+ \frac{\gamma^h}{\gamma}\int \frac{(1-\rho G s)(3\rho s - 1)}{(1-\rho s)^3} ds - \int \frac{(1-\rho G s)^2}{(1-\rho s)^3} ds\Bigg] \\ = \frac{2}{G^2}\Bigg[&\frac{\rho G^2 x(\rho x - 2) + 2(G-1)G(\rho x - 1)\log(\rho x - 1) + 2G - 1}{\rho(\rho x - 1)} \\ &- \frac{\varDelta\rho}{\sigma^2}\frac{\rho G x\left[\rho(Gx - 4) + 2G\right] + 2(\rho - G)^2\log(1-\rho x)}{2\rho^3} + \left(\frac{\gamma^h}{\gamma}\right)^2 \frac{3 - 4\rho x}{2\rho(\rho x - 1)^2} \\ &+ \frac{\gamma^h}{\gamma}\frac{G(4 - 5\rho x) + 3G(\rho x - 1)^2\log(\rho x - 1) + 3\rho x - 2}{\rho(\rho x - 1)^2} + \frac{2G^2\log(\rho x - 1) - \frac{(G-1)[G(4\rho x - 3) - 1]}{(\rho x - 1)^2}}{2\rho}\Bigg].\end{aligned}$$

Let $H = m\rho^h + \rho$, we have

$$\mu_c(x) = \sigma^2 + \varDelta\rho(1-\rho x) + \sigma^2 \frac{A_0 x^2 + A_1 x + A_2}{(1-\rho x)(Hx)^2} \text{ and } \sigma_c^2(x) = \sigma^2\left(\frac{1-Hx}{Hx}\right)^2,$$

which implies that:

$$\begin{aligned}\frac{2\mu_c(x)}{\sigma_c^2(x)} &= 2\left(\frac{Hx}{1-Hx}\right)^2 + 2\frac{\varDelta\rho}{\sigma^2}(1-\rho x)\left(\frac{Hx}{1-Hx}\right)^2 \\ &\quad + \frac{2A_0x^2}{(1-\rho x)(1-Hx)^2} + \frac{2A_1 x}{(1-\rho x)(1-Hx)^2} + \frac{2A_2}{(1-\rho x)(1-Hx)^2}\end{aligned}$$

and

$$
\begin{aligned}
\int \frac{2\mu_c(x)}{\sigma_c^2(x)} dx = & \frac{2\left[Hx + \frac{1}{1-Hx} + 2\log(1-Hx)\right]}{H} \\
& - \frac{2\Delta\rho}{\sigma^2} \frac{H^2\rho x^2 - 2Hx(H-2\rho) + \frac{2(H-\rho)}{Hx-1} + 2(3\rho - 2H)\log(1-Hx)}{H^2} \\
& - 2A_0 \frac{\frac{H-\rho}{H^2(Hx-1)} + \frac{(\rho-2H)\log(1-Hx)}{H^2} + \frac{\log(1-\rho x)}{\rho}}{(H-\rho)^2} - 2A_1 \frac{\frac{H-\rho}{H(Hx-1)} - \log(1-Hx) + \log(1-\rho x)}{(H-\rho)^2} \\
& + 2A_2 \frac{\rho(Hx-1)\log(1-Hx) + (\rho - H\rho x)\log(1-\rho x) - H + \rho}{(H-\rho)^2(Hx-1)}.
\end{aligned}
$$