Ignorance, Pervasive Uncertainty, and Household Finance*

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Abstract

This paper studies how the two types of uncertainty due to ignorance, parameter and model uncertainty, jointly affect strategic consumption-portfolio rules, precautionary savings, and welfare. We incorporate these two types of uncertainty into a recursive utility version of a canonical Merton (1971) model with uninsurable labor income and unknown income growth, and derive analytical solutions and testable implications. We show that the interaction between the two types of uncertainty plays a key role in determining the demand for precautionary savings and risky assets. We derive formulas to evaluate both marginal and total welfare costs of ignorance-induced uncertainty and show they are significant for plausible parameter values.

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1. Introduction

Most macroeconomic and financial models assume agents have a good understanding of the economic models they use to make optimal decisions. However, there is plenty of evidence that ordinary investors may be ignorant about certain aspects of the economic model, including the structure of the model, the parameters, and the current state of the model economy that their decisions are based on. For example, some households may not know the basics of risk diversification when making financial decisions (van Rooija et al., 2011);[1] some people may lack financial literacy, meaning that they do not have the necessary skills and knowledge to make informed and effective investment decisions (Lusardi and Mitchell, 2014); some investors may have incomplete information about the investment opportunity set (Brennen, 1998); and some individuals may not have full information about their own income growth (Guvenen, 2007). Such ignorance generates pervasive uncertainty for agents when they make economic and financial decisions.

Hansen and Sargent (2015) propose that using ignorance is a useful way to model different types of uncertainty by specifying the details that the decision maker is ignorant about. They use a simple Friedman one-equation tracking model to illustrate this idea. They mainly discussed two types of ignorance: (i) the agent is ignorant about the conditional distribution of the state variable in the next period and (ii) the agent is ignorant about the probability distribution of one key parameter (i.e., the response coefficient in their paper) in an otherwise fully trusted model. In other words, the first type of ignorance represents *model uncertainty* (or MU) as the agent does not know the distribution of shocks, while the second type of ignorance represents *parameter uncertainty* (or PU) as the agent does not know model parameters. (When parameters are stochastic, we may also view parameter uncertainty as state uncertainty.)[2]

Inspired by Hansen and Sargent (2015), in this paper we study how these two types of ignorance affect intertemporal consumption-saving and asset allocation decisions —a core focus of modern macroeconomics and finance. Our central goal is to provide a *unified* framework to study how these two types of ignorance (or the two types of uncertainty induced by ignorance: parameter and model uncertainty) interact with each other to affect the optimal consumption-saving-portfolio decisions as well as the equilibrium welfare implications. We derive analytical solutions to separate different forces which determine consumption, precautionary saving, and asset allocation. In addition, our framework also includes some other important factors such as incomplete markets, which are proven to be important for determining consumption, saving, and asset allo-

[1]Van Rooija et al. use the De Nederlandsche Bank (DNB) Household Survey data to study the relationship between financial literacy and stock market participation, and find that financial literacy affects financial decision-making: Those with low literacy are much less likely to invest in stocks.

[2]As the title of their paper suggests, Hansen and Sargent (2015) consider four types of ignorance. The first type is Friedman's hypothesis in which the agent does not know the key parameter but knows its distribution. The last type is called "structured uncertainty," which is less related to what we study in this paper. What we discuss here is related to the first three types in their paper.

cations (Wang, 2003, 2009).

Specifically, we construct a continuous-time Merton (1971)-Wang (2009)-type model with uninsurable labor income and unknown income growth in which investors have recursive utility and face model uncertainty. We follow Hansen and Sargent (2007) to introduce model uncertainty by incorporating the preference for robustness (RB). In robust control problems, agents are concerned about the possibility that their true model is misspecified in a manner that is difficult to detect statistically; consequently, they make their decisions as though the subjective distribution over shocks was chosen by an evil agent to minimize their utility.[3] In our model economy, investors not only have incomplete information about income growth but are also concerned about the model misspecification. Compared with the full-information rational expectations (FI-RE) case in which income growth is known, parameter uncertainty due to unknown income growth creates an additional demand for robustness. In our recursive utility framework, we also disentangle two distinct aspects of preferences: the agent's elasticity of intertemporal substitution (the EIS, attitudes towards variation in consumption across time) and the coefficient of absolute risk aversion (the CARA, attitudes toward variation in consumption across states), which are shown to have different roles in driving consumption-saving and portfolio choice decisions.[4] As will be explained below, our model delivers not only rich theoretical results but also testable implications.

This paper has four main findings and contributions. First, we provide analytical solutions to this rich framework with both types of uncertainty (PU and MU) to study strategic consumption-portfolio rules in the presence of uninsurable labor income. Using the closed-forms solution, we are able to inspect the exact mechanism through which these two types of induced uncertainty interact and affect the demand for risky assets and precautionary saving.[5] Specifically, we find that the precautionary saving demand and the strategic asset allocation are mainly affected by the *effective* coefficient of risk-uncertainty aversion ($\widetilde{\gamma}$) that is determined by the interaction between the CARA (γ), the EIS (ψ), and the degree of RB (ϑ) via the formula: $\widetilde{\gamma} = \gamma + \vartheta/\psi$. This expression clearly shows that both risk aversion and intertemporal substitution play roles in determining the amount of precautionary savings and the optimal share invested in the risky asset, but without model uncertainty, only risk aversion matters in determining these two demands.[6]

[3]There are three main ways to model ambiguity and robustness in the literature: the multiple priors model, the smooth ambiguity model, and the robust control/filtering model (Hansen and Sargent 2007). In this paper, we follow along the lines of Hansen and Sargent to introduce robustness and model uncertainty into our model.

[4]Constant-relative-risk-aversion (CRRA) utility functions are more common in macroeconomics, mainly due to balanced-growth requirements. CRRA utility would greatly complicate our analysis because the intertemporal consumption model with CRRA utility and stochastic labor income has no explicit solution and leads to non-linear consumption rules. See Cagetti et al. (2002) and Kasa and Lei (2018) for applications of RB in continuous-time models with CRRA utility.

[5]Hansen et al. (1999), Maenhout (2004), Ju and Miao (2012), Chen et al. (2014), and Luo and Young (2016) examine how model uncertainty and ambiguity affect portfolio choices and/or asset prices.

[6]The interaction between the EIS and RB in the expression for $\widetilde{\gamma}$ is due to two facts: (i) the value function in our benchmark model is a function of the EIS and (ii) the RB parameter is normalized by the value function (so it converts the relative entropy to units of utility and makes it consistent with the units of the expected future value function

Second, we use our analytical results (as summarized by Proposition 2 in Section 4) to separate different forces such as MU, PU, and the interaction between MU and PU in determining precautionary saving and the demand for risky assets, which represent the two core parts of saving and portfolio-choice decisions. We show that the share of precautionary saving due to MU and PU together is very large at reasonable levels of model uncertainty. For example, when the degree of model uncertainty (as measured by the detection error probability (DEP)) equals 0.2, MU and PU together can account for about 80% of total precautionary saving for plausible parameter values.[7] Out of this, about 46% comes from model uncertainty through the robust control channel and 54% comes from MU through robust filtering channel (or the interaction between PU and MU), while PU itself slightly *reduces* precautionary saving. We theoretically prove and explain why PU has such a negative impact on precautionary saving. In general, we show that the importance of the interaction between MU and PU increases with the robustness parameter, highlighting the importance of modeling the interaction between these two types of uncertainty when studying consumption-saving-portfolio choice problems. To the best of our knowledge, our paper is the first to provide a unified framework with analytical solutions to separate these different forces.

On the demand for risky assets, we show that its total demand decreases with the amount of model uncertainty due to robustness, and can be decomposed into three components: (i) the traditional speculation demand, (ii) the learning-induced hedging demand, and (iii) the income-hedging demand. We find that the first and second components are determined by the robust control and robust filtering channels respectively, while the third component is independent of both channels. In addition, this decline is largely driven by the increase in the degree of robustness through the robust control channel, though the relative importance of the robust filtering channel increases with the degree of robustness.

Third, we quantitatively test the model implications on risky-asset holding using household-level data . In particular, we conduct a calibration exercise to match the observed risky-asset holdings by different educational groups in the data. The calibrated results suggest that less educated households face more model uncertainty measured by the detection error probability, a commonly used statistical tool to measure the amount of model uncertainty in the robustness literature.[8] This result is consistent with the recent empirical finding using survey data by Lusardi and Mitchell (2014) which show that highly educated people are usually more financial literate than low educated people. In addition, we show that the interaction between the two types of uncertainty is the key to explain the data. In particular, if we shut down the model uncertainty channel, the model with only parameter uncertainty cannot explain the observed patterns of asset holdings.

evaluated with the distorted model). See Section 3 for a detailed discussion on this issue.

[7]DEP equal to 0.2 means that there is a probability of 0.2 that the reference model cannot be distinguished from the distorted model based on a likelihood ratio test.

[8]As we will explain later, a *higher* detection error probability means it is more difficult to distinguish the distorted model from the approximating model, or, the agent faces *less* model uncertainty.

Fourth, we show that the welfare cost of ignorance in general equilibrium can be sizable.[9] We provide formulas to evaluate both the marginal and the total welfare costs due to the two types of ignorance-induced uncertainties. Specifically, when $\vartheta = 2$ (or DEP=0.25), a 10% increase in ϑ leads to a welfare cost equivalent to about 2% percent reduction in initial consumption. In addition, this welfare loss increases with the amount of parameter uncertainty. Furthermore, if we remove all uncertainty in the model, the welfare gains could be as large as 23% of initial consumption.

The remainder of this paper is organized as follows. Section 2 provides a literature review. Section 3 describes our model setup, introducing key elements step by step. Section 4 presents key theoretical results. Section 5 presents quantitative results. Section 6 examines the welfare implications of ignorance. Section 7 concludes.

2. Literature Review

Our paper is related to two broad branches of literature. First, our paper is related to the broad literature studying consumption-saving and portfolio choices. Recent empirical studies on household portfolios in the U.S. and major European countries have stimulated research in allowing for portfolio choice between risky and risk-free financial assets when households receive labor income and have the precautionary saving motive. The empirical research on household portfolios documents both the increasing stock market participation rate in the U.S. and Europe and the importance of the precautionary saving motive for portfolio choice (see Guiso et al., 2002).[10] Some recent theoretical studies have also addressed the importance of parameter uncertainty or model uncertainty in affecting agents' optimal consumption and portfolio rules. For example, Brennen (1998) shows that the uncertainty about the mean return on the risky asset has a significant effect on the portfolio decision of a long-term investor. Maenhout (2004) explores how model uncertainty due to a preference for robustness affects optimal portfolio choice and the equilibrium equity premium. Wang (2009) finds that incomplete information about labor income growth can significantly affect optimal consumption-saving and asset allocation.

Second, on the modeling strategy, our paper is related to a fast growing literature on modeling induced uncertainty including both model uncertainty and parameter uncertainty. Besides Hansen and Sargent (2015), this paper is also closely related to Gennotte (1986), Maenhout (2004), Garlappi et al. (2007), Wang (2009), Collin-Dufresne et al. (2016), and Luo (2017). Maenhout (2004) explores how model uncertainty due to a preference for robustness reduces the demand for the risky asset and increases the equilibrium equity premium. Garlappi et al. (2007) examines how allowing for the possibility of multiple priors about the estimated expected returns affects optimal asset

[9]In Section 6 and the online appendix, we show that a unique general equilibrium under MU and PU can be constructed in the vein of Huggett (1993) and Wang (2003). (Wang (2003) constructs a general equilibrium under FI-RE in the same Bewley-Huggett type model economy with the CARA utility that we study.)

[10]See Campbell and Viceira (2002) for a textbook treatment on the studies of households' strategic asset allocation.

weights in a static mean-variance portfolio model. Wang (2009) studies the effects of incomplete information about the mean income growth on a consumer's consumption/saving and portfolio choice in an incomplete-market economy. Collin-Dufresne et al. (2016) study general equilibrium models with unknown parameters governing long-run growth and rare events, and show that parameter learning can generate quantitatively significant macroeconomic risks that help explain the existing asset pricing puzzles. Luo (2017) considers state uncertainty (uncertainty about the value of total wealth) within an expected utility partial equilibrium model, and finds that state uncertainty does not play an important role in determining strategic asset allocation unless the investors face very tight information-processing constraints.

Many empirical and experimental studies have repeatedly supported that individual agents are ambiguity averse (or have robustness preferences). For example, Ahn et al. (2014) use a rich experiment data set to estimate a portfolio-choice model and found that about 40% of subjects display either statistically significant pessimism or ambiguity aversion. Bhandari et al. (2019) identify ambiguity shocks using survey data, and show that in the data, the ambiguity shocks are an important source of variation in labor market variables.

However, our paper is also significantly different from the above papers. Unlike Maenhout (2004), our paper explores how the interaction of model uncertainty and parameter uncertainty affects the strategic consumption/saving-portfolio decisions in the presence of uninsurable labor income and unknown income growth. The model presented in this paper can therefore be used to study the relationship between the labor income risk and the stockholding behavior. Unlike Wang (2009) and Collin-Dufresne et al. (2016), this paper considers more general concepts of ignorance and induced uncertainty. We consider not only parameter uncertainty but also model uncertainty due to robustness. Another key difference between our paper and Wang (2009) is that rather than considering the discrete Markovian unknown income trend process as in Wang (2009), this paper considers a continuous Gaussian unknown process that can be estimated using PSID data and makes the general equilibrium welfare analysis tractable. In addition, unlike Garlappi et al. (2007), this paper focuses on incomplete information about the mean income growth, rather than incomplete information about the risk premium, and studies how this type of incomplete information affects the robust consumption and portfolio rules in an intertemporal setting. The key difference between this paper and Luo (2017) is that this paper focuses on examining parameter uncertainty that is more difficult to learn than the state uncertainty discussed in Luo (2017), and examines how it interacts with model uncertainty within a recursive utility general equilibrium framework in which both types of uncertainty interact with intertemporal substitution and risk aversion.

3. The Model Setup

In this section, we lay out our continuous-time consumption-portfolio choice model with recursive utility and two types of ignorance. To facilitate understanding of the key structure of the model, we will introduce each of the key elements one by one, starting with specifications of the labor income and investment opportunity set, followed by the description of the information set, then the recursive utility preference, and finally introducing the model uncertainty due to robustness.

As an overview, our model is a continuous-time recursive utility version of the Merton-type model (1971) with uninsurable labor income and unknown income growth. Specifically, we generalize the Wang (2009) model in the following three aspects: (i) rather than using the expected utility specification, we adopt a recursive utility specification; (ii) to better explore the importance of pervasive uncertainty due to ignorance in investors' financial decision-making problem, we not only consider parameter/state uncertainty due to unknown income growth, but also consider model uncertainty due to a preference for robustness; and (iii) instead of adopting the discrete-state Markovian income growth specification, here we consider a continuous-state Gaussian income growth specification, which can be estimated using the PSID data and can help explore the general equilibrium welfare implications of different types of ignorance. In summary, the typical investor in our model economy has recursive utility and makes strategic consumption-saving-asset allocation decisions with pervasive uncertainty. Finally, we assume that the investors can access two financial assets: one risk-free asset and one risky asset.

3.1. Specifications of Labor Income and Investment Opportunity Set

Labor income (y_t) is assumed to follow a continuous-time Ornstein-Uhlenbeck (OU) process:

$$dy_t = (\mu_t - \rho y_t)\, dt + \sigma_y dB_{y,t}, \tag{1}$$

where σ_y is the unconditional volatility of the income change over an incremental unit of time, the persistence coefficient ρ governs the speed of convergence or divergence from the steady state, $B_{y,t}$ is a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$, and μ_t is the *unobservable* income growth rate. Specifically, we assume that μ_t follows a mean-reverting Ornstein-Uhlenbeck process:

$$d\mu_t = \lambda\,(\overline{\mu} - \mu_t)\, dt + \sigma_\mu dB_{\mu,t}, \tag{2}$$

where $B_{\mu,t}$ is a standard Brownian motion defined over the complete probability space, λ and $\overline{\mu}$ are positive constants, and the correlation between $dB_{y,t}$ and $dB_{\mu,t}$ is $\rho_{y\mu} \in [-1, 1]$. (That is, $\mathbb{E}\left[dB_{y,t} dB_{\mu,t}\right] = \rho_{y\mu} dt$.) In other words, in this setting, both the income growth rate and actual income are stochastic and risky. Since μ_t is unknown to investors, the investors need to estimate it using their observations of the realized labor income. In the traditional signal extraction models, the typical investor estimates the conditional distribution of the true income growth rate and then represents the investor's original optimizing problem as a Markovian one. If we assume that the loss function in our model is the mean square error (MSE) due to incomplete information, then given a Gaussian prior, finding the posterior distribution of the income growth rate becomes a standard Kalman-Bucy filtering problem. However, given that we assume that households have a preference for robustness, they might not only consider a robust control problem but also consider a corresponding robust filtering problem. (We will discuss the robust filtering problem after introducing model uncertainty due to robustness in Section 3.3.)

Households (investors) in our model economy can invest in both a risk-free asset with a constant interest rate r and a risky asset (i.e., the market portfolio) with a risky return r_t^e. The instantaneous return dr_t^e of the risky market portfolio over dt is given by:

$$dr_t^e = (r + \pi)\, dt + \sigma_e dB_{e,t}, \tag{3}$$

where π is the market risk premium, σ_e is the standard deviation of the market return, and $B_{e,t}$ is a standard Brownian motion defined on $(\Omega, \mathcal{F}_t, \mathbb{P})$ and is correlated with the Brownian motion, $B_{y,t}$. Let ρ_{ye} be the contemporaneous correlation between the labor income process and the return of the risky asset. When $\rho_{ye} = 0$, the labor income risk is purely idiosyncratic and is uncorrelated with the risky market return; when $\rho_{ye} = 1$, the labor income risk is perfectly correlated with the risky market return. The typical household's financial wealth evolution is given by

$$dw_t = (rw_t + y_t - c_t)\, dt + \alpha_t\, (\pi dt + \sigma_e dB_{e,t})\,, \tag{4}$$

where α_t denotes the amount of wealth that the investor allocates to the market portfolio at time t.

3.2. Recursive Exponential Utility

In this paper, we assume that investors in our model economy have a recursive utility preference of the Kreps-Porteus/Epstein-Zin type, and can disentangle the degree of risk aversion from the elasticity of intertemporal substitution. Although the expected utility model has many attractive features, it implies that the agent's elasticity of intertemporal substitution is the reciprocal of the coefficient of risk aversion. Conceptually, however, risk aversion and intertemporal substitution capture two distinct aspects of the decision-making problem. Specifically, for every stochastic consumption-portfolio stream, $\{c_t, \alpha_t\}_{t=0}^{\infty}$, the utility stream, $\{V(U_t)\}_{t=0}^{\infty}$, is recursively defined as follows:

$$V(U_t) = \left(1 - e^{-\beta\Delta t}\right) V(c_t) + e^{-\beta\Delta t} V\left(\mathbb{CE}_t\left[U_{t+\Delta t}\right]\right), \tag{5}$$

where Δt is time interval, $\beta > 0$ is the agent's subjective discount rate, $V(c_t) = (-\psi)\exp(-c_t/\psi)$, $V(U_t) = (-\psi)\exp(-U_t/\psi)$,

$$\mathbb{CE}_t[U_{t+\Delta t}] = G^{-1}(\mathbb{E}_t[G(U_{t+\Delta t})]),$$

is the certainty equivalent of U_{t+1} conditional on the period t information, and $G(U_{t+\Delta t}) = -\exp(-\gamma U_{t+\Delta t})/\gamma$. In (5), $\psi > 0$ governs the elasticity of intertemporal substitution (EIS), while $\gamma > 0$ governs the coefficient of absolute risk aversion (CARA).[11] In other words, a high value of ψ corresponds to a strong willingness to substitute consumption over time, and a high value of γ implies a high degree of risk aversion. It is well-known that the CARA (or negative exponential) utility models including our recursive exponential utility (REU) do not rule out negative consumption because the marginal utility will not approach infinity as consumption converges to zero. We do not impose a non-negativity consumption constraint explicitly in this paper because a closed-form solution to the problem would not be available if that constraint were binding. Note that when $\psi = 1/\gamma$, the functions V and G are the same and the recursive utility reduces to the standard time-separable expected utility function used in Caballero (1990) and Wang (2003, 2009). In addition, $\psi = 1/\gamma$ also implies that the investor is indifferent about the time at which uncertainty is resolved.[12]

3.3. Incorporating Model Uncertainty Due to Robustness

As we mentioned above, the loss function in the model in which investors have a preference for robustness might not still be the mean square error (MSE) due to incomplete information; consequently, given a Gaussian prior, finding the posterior distribution of μ_t may *not* be a standard Kalman-Bucy filtering problem. In other words, in the problem studied in this paper, robustness may not only be applied to the control problem (robust control), but may also be applied to the filtering problem (robust filtering). Hansen and Sargent (2007) argue that the simplest version of robustness considers the question of how to make optimal decisions when the decision-maker does not know the true probability model that generates the data. The main goal of introducing robustness is to design optimal policies that not only work well when the reference model governing the evolution of the state variables is the true model, but also performs reasonably well when the true economy is governed by the distorted model. In this paper, we follow Kasa (2006) and Hansen and Sargent (2007, Chapter 17), and consider a situation in which the households pursue *a robust Kalman gain* and faces the commitment on the part of the minimizing agent to previous distortions. (Note that Hansen and Sargent (2007) also discussed robust filtering without commitment.) Specifically, to solve this robust control and filtering problem, we adopt a two-stage procedure that can

[11] It is well known that the CARA utility specification is tractable for deriving the consumption function or optimal consumption-portfolio rules in different settings. See Merton (1971), Caballero (1990), Calvet (2001), Wang (2003, 2009), Angeletos and Calvet (2006), and Luo (2017).

[12] Note that the investor prefers early resolution of uncertainty if $\gamma > 1/\psi$ and prefers late resolution if $\gamma < 1/\psi$.

decompose the original incomplete-information optimal problem into two sub-problems:[13]

1. Robust filtering problem. In the first stage, we assume that the investors solve a robust filtering problem. The main idea of robust filtering is the same as the standard filtering problem in which investors want to minimize the expected loss function determined by the difference between the estimated state and the true state. The new feature is that in the robust filtering problem, investors have a preference for robustness, which means they are concerned that the state transition and observation equations are misspecified. Consequently, the optimal Kalman gain is now affected by the preference for robustness.

2. Robust control problem. In the second stage, treating the perceived unobservable state as a underlying state variable, the investors who are concerned about the model misspecification solve the robust consumption-portfolio choice model. We then verify that the corresponding value function leads to the same loss function we use in Stage 1.

It is worth noting that when the agent cannot observe μ_t perfectly, his optimization problem is not recursive. However, adopting the above two-stage procedure helps convert the original non-recursive optimization problem into a recursive formulation because the dynamics of the estimated state contains the same information as the original non-recursive incomplete-information model.

3.3.1. Robust Filtering

Following Kasa (2006) and Hansen and Sargent (2007, Chapter 17), we first consider the filtering problem. Specifically, we consider a situation in which the investor who is concerned about the model pursues a robust Kalman gain. To obtain a robust Kalman filter gain, we first use relative entropy to measure model uncertainty, and then use the Girsanov theorem to parameterize the distorted model. Here, we use $\mathbb{P}$ and $\mathbb{Q}$ to denote the probability measures of the approximating and distorted models, respectively.

Following the robust filtering literature, applying the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ to the state transition and observation equations, (2) and (1), yields the following distorted filtering model:

$$d\mu_t = \left[\lambda\left(\overline{\mu} - \mu_t\right) + \sigma_\mu v_{1,t}\right] dt + \sigma_\mu d\widetilde{B}_{\mu,t}, \tag{6}$$

$$dy_t = \left(\mu_t - \rho y_t + \sigma_y v_{2,t}\right) dt + \sigma_y d\widetilde{B}_{y,t}, \tag{7}$$

where $\widetilde{B}_{\mu,t}$ and $\widetilde{B}_{y,t}$ are Wiener processes that are related to the corresponding approximating processes:

$$\widetilde{B}_{\mu,t} = B_{\mu,t} - \int_0^t v_{1,s} ds \text{ and } \widetilde{B}_{y,t} = B_{y,t} - \int_0^t v_{2,s} ds$$

[13] As in Wang (2004, 2009), the separation principle also applies to our robust filtering and control problem given the CARA-Gaussian specification.

and $v_{1,t}$ and $v_{2,t}$ are distortions to the conditional means of the two shocks, $\widetilde{B}_{\mu,t}$ and $\widetilde{B}_{y,t}$, respectively. As shown in Pan and Basar (1996); Ugrinovskii and Petersen (2002); and Kasa (2006), a robust filter can be characterized by the following dynamic zero-sum game:

$$L_t = \inf_{\{m_j\}} \sup_{\{\mathbb{Q}\}} \left\{ \limsup_{T\to\infty} \mathbb{E}^{\mathbb{Q}} [F] - \vartheta^{-1} \mathbb{H}_\infty (\mathbb{Q}|\mathbb{P}) \right\}, \tag{8}$$

subject to (6) and (7), where $m_j = \mathbb{E}_j [\mu_j]$ is the conditional mean of μ_j, $F \equiv T^{-1} \int_0^T (\mu_j - m_j)^2 dj$ is the loss function, and $\mathbb{H}_\infty$ is the relative entropy and is bounded from above:

$$H_\infty (\mathbb{Q}|\mathbb{P}) = \limsup_{T\to\infty} \mathbb{E}^{\mathbb{Q}} \left[\frac{1}{2T} \int_0^T (v_{1,t}^2 + v_{2,t}^2) \, dj \right] \le \eta_0, \tag{9}$$

where η_0 defines the set of models that the investor is considering, and ϑ^{-1} is the Lagrange multiplier on the relative entropy constraint, (9). In Appendices 8.1 and 8.2, we show that the loss function due to incomplete information in our REU setting is (approximately) quadratic. Following the literature, we use ϑ to measure the degree of robustness throughout the paper. As shown in Dai Pra et al. (1996), the entropy constrained robust filtering problem, (8), is equivalent with the following risk-sensitive filtering problem:[14]

$$\frac{1}{\vartheta} \log \left(\int \exp (\vartheta F(\mu, m)) \, d\mathbb{P} \right) = \sup_{Q} \left\{ \int F(\mu, m) \, d\mathbb{Q} - \vartheta^{-1} \mathbb{H}_\infty (\mathbb{Q}|\mathbb{P}) \right\}. \tag{10}$$

The following proposition summarizes the results for this robust filtering problem:

Proposition 1. *When $\vartheta \ge \sigma_\mu^2/(\sigma_\mu^2 + \lambda^2)$, there is a unique solution for the robust filtering problem, (10):*

$$dm_t = \lambda (\overline{\mu} - m_t) \, dt + K_t [dy_t - (m_t - \rho y_t) \, dt], \tag{11}$$

$$\frac{d\Sigma_t}{dt} = -2\lambda \Sigma_t + \sigma_\mu^2 - \left(\frac{1}{\sigma_y^2} - \vartheta \right) (\eta + \Sigma_t)^2, \tag{12}$$

where $m_t = \mathbb{E}_t [\mu_t]$ and $\Sigma_t = \mathbb{E}_t \left[(\mu_t - m_t)^2 \right]$ is the conditional mean and variance of μ_t, respectively,

$$K_t = (\eta + \Sigma_t) / \sigma_y^2 \tag{13}$$

is the Kalman gain, and $\eta = \rho_{y\mu} \sigma_y \sigma_\mu$,. In the steady state, the conditional variance converges to:

$$\Sigma^* = \frac{-\sigma_y^2 \lambda + \sigma_y^2 \sqrt{\widetilde{\lambda}^2 + \left(1 - \rho_{y\mu}^2\right) \left(\sigma_\mu^2/\sigma_y^2\right) - \vartheta \left(2\lambda\eta + \sigma_\mu^2\right)}}{1 - \vartheta \sigma_y^2} - \eta \ge 0, \tag{14}$$

[14] See Online Appendix A for a proof.

where $\widetilde{\lambda} = \lambda + \eta/\sigma_y^2$.

Proof. The right-hand side of (10) is an entropy constrained filtering problem in terms of the scaled quadratic objective function F, while the left-hand side is a risk-sensitive filtering problem in terms of the same function F, with risk-sensitivity parameter, ϑ. The solution of this risk-sensitive filtering problem is just a special case of Pan and Basar (1996)'s Theorem 3. ■

Using the expected mean m_t, we can now rewrite the income process as:

$$dy_t = (m_t - \rho y_t)\, dt + \sigma_y dB_{m,t}, \tag{15}$$

where $dB_{m,t} = dB_{y,t} + \left(\frac{\mu_t - m_t}{\sigma_y}\right) dt$ is the normalized unanticipated innovation of the income growth process and a standard Brownian motion with respect to the investor's filtration. Substituting it into the estimated state updating equation yields:

$$dm_t = \lambda\left(\overline{\mu} - m_t\right) dt + \sigma_m dB_{m,t}, \tag{16}$$

where:

$$\sigma_m \equiv \sigma_y K = \frac{\eta + \Sigma^*}{\sigma_y} = \frac{-\sigma_y\lambda + \sigma_y\sqrt{\widetilde{\lambda}^2 + \left(1 - \rho_{y\mu}^2\right)\left(\sigma_\mu^2/\sigma_y^2\right) - \vartheta\left(2\lambda\eta + \sigma_\mu^2\right)}}{1 - \vartheta\sigma_y^2} \tag{17}$$

is the diffusion coefficient (i.e., the instantaneous standard deviation of m_t) and Σ^* is given in (14). It should be noted that the information structure generated by $\{m_0, B_{m,s}, s \in [0,t]\}$ is the same as the one generated by $\{y_s, s \in [0,t]\}$. It is worth noting that when $\vartheta = 0$, the above robust filtering problem reduces to the traditional optimal filtering problem in which:

$$\Sigma^* = -\sigma_y^2\widetilde{\lambda} + \sigma_y^2\sqrt{\widetilde{\lambda}^2 + \frac{\left(1 - \rho_{y\mu}^2\right)\sigma_\mu^2}{\sigma_y^2}} \text{ and } \sigma_m = -\sigma_y\lambda + \sigma_y\sqrt{\widetilde{\lambda}^2 + \frac{\left(1 - \rho_{y\mu}^2\right)\sigma_\mu^2}{\sigma_y^2}}. \tag{18}$$

Note that when $\rho_{y\mu} = 1$, the optimal filtering model reduces to the full-information model in which μ_t is fully observable, $\Sigma^* = 0$, and σ_m converges to σ_μ.[15] As mentioned before, in this paper we treat $\sigma_\mu > 0$ as fundamental uncertainty, and treat the noise-to-signal ratio, σ_y^2/σ_μ^2, as parameter uncertainty. Given $\rho_{y\mu}$, the learning mechanism is determined by the noise-to-signal ratio.

[15]When $\vartheta = 0$, it is straightforward to show that:

$$\sigma_m = \sigma_y\lambda + \sigma_y\sqrt{\lambda^2 + 2\lambda\frac{\rho_{y\mu}\sigma_\mu}{\sigma_y} + \left(\frac{\sigma_\mu}{\sigma_y}\right)^2} \leq \sigma_\mu$$

when $\rho_{y\mu} \leq 1$. Throughout this paper, we rule out the possibility that $\sigma_\mu = 0$ because the estimated value of this parameter using the PSID is significantly positive.

From (13), (14), and (17), it is straightforward to show that for different values of σ_y/σ_μ,

$$\frac{\partial \Sigma^*}{\partial \vartheta} > 0, \frac{\partial \sigma_m}{\partial \vartheta} > 0, \text{ and } \frac{\partial K}{\partial \vartheta} > 0.$$

That is, the steady state conditional variance (Σ^*), the diffusion coefficient (σ_m), and the robust Kalman gain (K) are increasing with the preference for pursuing a robust Kalman filter. In Section 5, we will quantitatively examine the relative importance of robust control and filtering in determining precautionary savings and strategic asset allocation after using the US data to estimate the joint y and μ process.

3.3.2. Robust Control

To introduce robust control into our model proposed above, we follow the continuous-time methodology proposed by Anderson et al. (2003) and adopted in Maenhout (2004) to assume that investors are concerned about the model misspecifications and take Equations (4), (15), and (16) as the approximating model. They accept the approximating model as the best approximating model, but are still concerned that it is misspecified. They therefore want to consider a range of models (i.e., the distorted model) surrounding the approximating model when computing the continuation payoff.

More specifically, the corresponding distorting model can thus be obtained by adding an endogenous distortion $v(s_t)$ to the approximating model:

$$ds_t = (\Lambda + \Sigma \cdot v_t)\, dt + \sigma \cdot dB_t, \tag{19}$$

where $s_t = \begin{bmatrix} w_t & y_t & m_t \end{bmatrix}^T$, $ds_t = \begin{bmatrix} dw_t & dy_t & dm_t \end{bmatrix}^T$, $\Lambda = \begin{bmatrix} rw_t + y_t - c_t + \alpha_t\pi & m_t - \rho y_t & \lambda(\overline{\mu} - m_t) \end{bmatrix}^T$, $v_t = \begin{bmatrix} v_{1,t} & v_{2,t} & v_{3,t} \end{bmatrix}^T$, $dB_t = \begin{bmatrix} dB_{e,t} & dB_{i,t} \end{bmatrix}^T$,[16] $\sigma = \begin{bmatrix} \alpha_t\sigma_e & 0 \\ \rho_{ye}\sigma_y & \sqrt{1-\rho_{ye}^2}\sigma_y \\ \rho_{ye}\sigma_m & \sqrt{1-\rho_{ye}^2}\sigma_m \end{bmatrix}$, and $\Sigma \equiv \sigma\sigma^T = \begin{bmatrix} \alpha_t^2\sigma_e^2 & \rho_{ye}\sigma_y\alpha_t\sigma_e & \rho_{ye}\alpha_t\sigma_e\sigma_m \\ \rho_{ye}\sigma_y\alpha_t\sigma_e & \sigma_y^2 & \sigma_y\sigma_m \\ \rho_{ye}\alpha_t\sigma_e\sigma_m & \sigma_y\sigma_m & \sigma_m^2 \end{bmatrix}$.

Under RB, the HJB can thus be written as:

$$\beta V(J_t) = \sup_{\{c_t,\alpha_t\}} \inf_{v_t} \left\{ \beta V(c_t) + \mathcal{D}V(J_t) + \frac{1}{\vartheta_t}\mathcal{H} \right\}, \tag{20}$$

[16] To obtain independent Brownian motions, we apply a Cholesky decomposition to $dB_{m,t}$: $dB_{m,t} = \rho_{ye} dB_{e,t} + \sqrt{1-\rho_{ye}^2} dB_{i,t}$, where $B_{i,t}$ is a standard Brownian motion and independent of $B_{e,t}$.

where

$$\mathcal{D}V(J_t) = V'(J_t)\left((\partial J)^T \cdot E[ds_t] + (\partial J)^T \cdot \Sigma \cdot v_t - \frac{\gamma}{2}\left[(\partial J)^T \cdot \Sigma \cdot \partial J\right]\right). \tag{21}$$

Here $\partial J = \begin{bmatrix} J_w & J_y & J_m \end{bmatrix}^T$ and $J_t = -\alpha_0 - \alpha_1 w_t - \alpha_2 y_t - \alpha_3 m_t$, where α_0, α_1, α_2, and α_3 are undetermined coefficients. The first two terms in (21) are the expected continuation payoff when the state variable follows (19), i.e., the alternative model based on drift distortion $v(s_t)$, $\mathcal{H} = \left(v_t^T \cdot \Sigma \cdot v_t\right)/2$ is the relative entropy or the expected log likelihood ratio between the distorted and the approximating models and measures the distance between the two models, and $1/\vartheta_t$ is the weight on the entropy penalty term.[17] The last term, $\mathcal{H}/\vartheta_t$, in (20) quantifies the penalty due to RB.

In (20), the drift adjustment $v(s_t)$ is chosen to minimize the sum of (i) the expected continuation payoff adjusted to reflect the additional drift component in (19) ($(\partial J)^T \cdot \Sigma \cdot v_t$) and (ii) an entropy penalty ($\mathcal{H}/\vartheta_t$):

$$\inf_{v}\left[V'(J_t)(\partial J)^T \cdot \Sigma \cdot v_t + \frac{1}{\vartheta_t}\mathcal{H}\right], \tag{22}$$

where ϑ_t is state-dependent as in Maenhout (2004). The key reason for using a state-dependent $\vartheta_t\,(=\vartheta(s_t))$ is to assure the homotheticity or scale invariance of the decision problem in our CARA-Gaussian setting.[18] Solving first for the infimization part of (22) yields $v_t^* = -\vartheta_t V'(J_t)\,\partial J$, where $\vartheta_t = -\vartheta/V(J_t) > 0$ and ϑ is a constant (see Appendix 8.1 for the derivation).

Using a given detection error probability, we can easily calibrate the corresponding value of ϑ that affects the optimal consumption-portfolio rules. Substituting for v^* in (20) leads to the following HJB:

$$\beta V(J_t) = \sup_{\{c_t,\alpha_t\}}\left\{\beta V(c_t) + V'(J_t)\left((\partial J)^T \cdot E[ds_t] - \frac{1}{2}\widetilde{\gamma}\left[(\partial J)^T \cdot \Sigma \cdot \partial J\right]\right)\right\}, \tag{23}$$

where $\widetilde{\gamma} = \gamma + \vartheta/\psi$.

It is worth noting that we can follow Munk and Sørensen (2010) and Wang et al. (2016) to construct human wealth as the discount present value of the current and future labor incomes. Once we define the total wealth of the typical investor as the sum of financial wealth and human wealth, we can reduce the state space of our model to one dimension and write down the optimal investment and consumption policies for a Merton problem (without labor income) as a function of total wealth. Specifically, in our incomplete markets economy, there exists a unique stochastic

[17] Note that the $\vartheta_t = 0$ case corresponds to the standard expected utility case. This robustness specification is called the *multiplier (or penalty)* robust control problem.

[18] Note that the impact of robustness wears off if we assume that ϑ_t is constant. This is clear from the procedure of solving the robust HJB proposed. As argued in Maenhout (2004), here we can also define "$1/V(J_t)$" in the ϑ_t specification as a *normalization* factor that is introduced to convert the relative entropy to units of utility so that it is consistent with the units of the expected future value function evaluated with the distorted model.

discount factor ζ_t under the *minimal* martingale measure $\mathbb{Q}^m$ satisfying:

$$d\zeta_t = -\zeta_t \left(rdt + \frac{\rho_{ye}\pi}{\sigma_e} dB_{e,t} \right),$$

where $\zeta_0 = 1$. The human wealth under incomplete markets is then defined as follows:

$$\widetilde{h}(y_t, m_t) = \mathbb{E}_t^{\mathbb{P}} \left[\int_t^\infty \frac{\zeta_s}{\zeta_t} y_s ds \middle| \mathcal{F}_t^y \right] = \mathbb{E}_t^{\mathbb{Q}^m} \left[\int_t^\infty e^{-r(s-t)} y_s ds \middle| \mathcal{F}_t^y \right], \tag{24}$$

where $\mathbb{Q}^m$ is the risk-neutral probability measure with respect to $\mathbb{P}$. Using (1) and (16), the expression of $\widetilde{h}(y, m)$ can thus be simplified as:

$$\widetilde{h}(y_t, m_t) = \frac{y_t}{r+\rho} + \frac{m_t}{(r+\rho)(r+\lambda)} + \frac{1}{r(r+\rho)(r+\lambda)} \left\{ \lambda\bar{\mu} - \left[(r+\lambda)\sigma_y + \sigma_m \right] \frac{\rho_{ye}\pi}{\sigma_e} \right\}.$$

Using $a_t = w_t + \widetilde{h}_t$ to denote the total wealth, we have:

$$da_t = \left(ra_t + \alpha_t \pi + \frac{\rho_{ye}\pi\sigma_h}{\sigma_e} - c_t \right) dt + \alpha_t \sigma_e dB_{e,t} + \sigma_h dB_{m,t}, \tag{25}$$

where $\sigma_h = \left[(r+\lambda)\sigma_y + \sigma_m \right] / \left[(r+\rho)(r+\lambda) \right]$. This model can also be seen as a Merton problem, and delivers the same solution as obtained in our original model. (See Online Appendix B for the derivation.)

4. Theoretical Results and Implications

This section provides our theoretical results. We analytically solve the model and obtain the robustly strategic consumption-portfolio rules. Then we use the analytical solutions to separate the contribution from the two types of ignorance-induced uncertainty, MU and PU, in determining the key components in saving and portfolio-choice decisions. We finally explore how the induced uncertainty interacts with the discount rate and the interest rate and affects consumption dynamics.

4.1. Inspecting the Robust Consumption-Saving-Portfolio Rules

The following proposition summarizes the solution and shows how different forces jointly determine the consumption, saving, and portfolio choices:

Proposition 2. *Under robustness and unknown income trend, the decision rules for consumption and portfolio choices, as well as the value function are given by:*

(i) The optimal consumption rule is:

$$c_t^* = r(w_t + h_t + l_t) + \Psi + \Pi - \Gamma, \tag{26}$$

where

$h_t = \frac{1}{r+\rho}\left(y_t + \frac{\overline{\mu}}{r} - \frac{\pi\rho_{ye}\sigma_y}{r\sigma_e}\right)$ *is the risk-adjusted human wealth;*

$l_t = \frac{1}{(r+\rho)(r+\lambda)}\left(m_t - \overline{\mu} - \frac{\pi\rho_{ye}\sigma_m}{r\sigma_e}\right)$ *is the perceived risk-adjusted income trend;*

$\Psi = \left(\frac{\beta-r}{r}\right)\psi$ *measures the effects of impatience on consumption and saving;*

$\Pi = \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}$ *is the additional increase in the investor's certainty equivalent wealth due to the presence of the risky asset;*

$$\Gamma \equiv \frac{1}{2}r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)\left[\frac{\sigma_y}{r+\rho} + \frac{\sigma_m}{(r+\rho)(r+\lambda)}\right]^2 \tag{27}$$

is the investor's precautionary saving demand, with $\widetilde{\gamma} \equiv \gamma + \vartheta/\psi$ *being the effective coefficient of absolute risk-uncertainty aversion.*

(ii) The optimal portfolio rule is:

$$\alpha_t^* = \alpha_s + \alpha_l + \alpha_y, \tag{28}$$

where

$$\alpha_s = \frac{\pi}{r\widetilde{\gamma}\sigma_e^2} \tag{29}$$

is the standard speculation demand for the risky asset,

$$\alpha_l \equiv -\frac{\rho_{ye}\sigma_m}{\sigma_e(r+\rho)(r+\lambda)} \tag{30}$$

is the learning-induced hedging demand,

$$\alpha_y = -\frac{\rho_{ye}\sigma_y}{\sigma_e(r+\rho)} \tag{31}$$

is the labor income-hedging demand, and σ_m *is given by (17).*

(iii) The associated value function is

$$V(J_t) = (-\psi)\exp\left(\left[\alpha_0 - rw_t - \frac{r}{r+\rho}y_t - \frac{r}{(r+\rho)(r+\lambda)}m_t\right]/\psi\right), \tag{32}$$

where

$$\begin{aligned}\alpha_0 = {} & \left(1-\frac{\beta}{r}\right)\psi - \psi\ln\left(\frac{r}{\beta}\right) - \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2} + \frac{\pi\rho_{ye}}{\sigma_e}\left[\frac{\sigma_y}{(r+\rho)} + \frac{\sigma_m}{(r+\rho)(r+\lambda)}\right] \\ & - \frac{\lambda\overline{\mu}}{(r+\rho)(r+\lambda)} + \frac{1}{2}r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)\left[\frac{\sigma_y}{r+\rho} + \frac{\sigma_m}{(r+\rho)(r+\lambda)}\right]^2.\end{aligned} \tag{33}$$

Proof. See Appendix A for the derivation. ∎

We make several comments on this important proposition. First, the consumption rule (26) illustrates how the optimal consumption is influenced by different forces in this rich framework. In the standard permanent income hypothesis model, optimal consumption is determined by the annuity value of the lifetime wealth, $r(w_t + h_t)$. Other terms on the right hand side of (26) show the influence of the MU (via ϑ), PU (via $K = \sigma_m/\sigma_y$), incomplete market and their interactions on consumption. Specifically, l_t in the consumption function can be viewed as the investor's risk-adjusted certainty equivalent wealth due to the uncertainty about his income trend. Ψ captures the extra saving due to relative impatience. When the interest rate, r, is endogenously determined, this term also captures the effects of incomplete market on the equilibrium interest rate and savings. Π captures the extra savings when there are risky assets in the economy. Γ captures the effect of precautionary savings, which is jointly influenced by MU through ϑ and by PU through the Kalman gain, K. As the Kalman gain is also affected by the robustness parameter ϑ, it is easy to see that the preference for robustness (ϑ) affects the precautionary saving demand via two channels: (i) the direct channel (the robust control channel) and (ii) the indirect channel (the robust filtering channel) via the robust Kalman gain (K).[19] In addition, both the correlation between the equity return and labor income (ρ_{ye}) and the correlation between labor income and the mean growth rate of labor income ($\rho_{y\mu}$) interact with the volatility of the noisy signal (σ_y) and the volatility of the perceived mean (σ_m), and then affect the precautionary saving demand.

Second, our analytical solution allows us to conduct various decompositions to help understand how PU and MU influence precautionary savings. For example, to fully explore how parameter uncertainty due to unknown income growth affects the precautionary saving demand, we first shut down both the parameter uncertainty channel and the model uncertainty channel. In this case, the investor has complete information about the parameter and has no concerns about model specification, and the precautionary saving demand, Γ_0, can be written as:[20]

$$\Gamma_0 = \frac{1}{2} r\gamma \left(1 - \rho_{ye}^2\right) \left[\frac{\sigma_y}{r+\rho} + \frac{\sigma_\mu}{(r+\rho)(r+\lambda)}\right]^2. \tag{34}$$

When we only shut down the model uncertainty channel, the precautionary saving demand, $\Gamma_{PU,0}$, can be written as:

$$\Gamma_{PU,0} = \frac{1}{2} r\gamma \left(1 - \rho_{ye}^2\right) \left[\frac{\sigma_y}{r+\rho} + \frac{\sigma_{m,0}}{(r+\rho)(r+\lambda)}\right]^2, \tag{35}$$

where $\sigma_{m,0} = -\sigma_y \lambda + \sigma_y \sqrt{\widetilde{\lambda}^2 + \frac{\left(1-\rho_{y\mu}^2\right)\sigma_\mu^2}{\sigma_y^2}}$ is the diffusion coefficient in dm_t when $\vartheta = 0$. We can then define the amount of precautionary savings due to parameter uncertainty as:

$$\Gamma_{PU} = \Gamma_{PU,0} - \Gamma_0. \tag{36}$$

[19] In the next section, we will quantitatively show that the indirect channel has a significant impact on precautionary saving and asset allocation and is comparable to the direct channel.

[20] Note that here when $\rho_{y\mu} = 1$, $\sigma_m = \sigma_\mu$.

Given the expressions of Γ and $\Gamma_{PU,0}$, we define the amount of precautionary savings due to model uncertainty as:

$$\Gamma_{MU} = \Gamma - \Gamma_{PU,0}. \tag{37}$$

We then define

$$\Gamma_{PU,MU} = \Gamma_{PU} + \Gamma_{MU} = \Gamma - \Gamma_0 \tag{38}$$

as the additional demand for precautionary saving due to the interactions of model and parameter uncertainty (i.e., the ignorance of the unknown parameter and model specification).

It is worth noting that we can decompose the precautionary saving demand due to robustness and model uncertainty into: (i) the *robust control* component and (ii) the *robust filtering* component. Here we can use the amount of the robust filtering component to measure the interaction between parameter and model uncertainty. That is, the robust filtering part (denoted by $\Gamma_{MU,filtering}$) captures the joint effect of robustness and learning in determining precautionary saving. Specifically, we decompose Γ_{MU} as:

$$\Gamma_{MU} = \Gamma_{MU,control} + \Gamma_{MU,filtering}, \tag{39}$$

where

$$\Gamma_{MU,control} = \frac{1}{2} r \left(\widetilde{\gamma} - \gamma\right) \left(1 - \rho_{ye}^2\right) \left[\frac{\sigma_y}{r+\rho} + \frac{\sigma_{m,0}}{(r+\rho)(r+\lambda)}\right]^2 \tag{40}$$

measures the precautionary saving demand due to robust control. Note that here the robust control component is mainly determined by the increase in the effective coefficient of risk-uncertainty aversion from γ to $\widetilde{\gamma}$, while the robust filtering component is mainly determined by the diffusion coefficient, σ_m. In the next subsection, we will evaluate the relative importance of the additional demand due to parameter and model uncertainty after estimating the y_t and μ_t processes.

Third, expressions (28)-(30) clearly show how the interactions of parameter and model uncertainty affect the learning-induced hedging demand for the risky asset (α_l). As a reminder, from (29) we see that the standard speculation demand for the risky asset (α_s) only depends on the robust control channel and is irrelevant with the robust filtering part. In other words, this traditional demand for the risky asset only relies on the control part. In contrast, (30) shows that the learning-induced hedging demand (α_l) only depends on the robust filtering channel and is independent of the robust control channel. Note that we can view the robust filtering channel as the interaction between model and parameter uncertainty. In addition, (29) shows that the income-hedging demand for the risky asset (α_y) is independent of both parameter and model uncertainty.

Therefore, our analytical solutions allow us to exactly inspect the relative importance of the three components in the total demand for the risky asset. Specifically, using (29) and (31), the relative importance of the learning-induced hedging demand to the traditional speculation demand can be written as:

$$\Xi_{ls} \equiv \frac{|\alpha_l|}{\alpha_s} = \frac{r\rho_{ye}\sigma_e\widetilde{\gamma}\sigma_m}{\pi\left(r+\rho\right)\left(r+\lambda\right)}, \tag{41}$$

which clearly shows that both the robust control ($\widetilde{\gamma}$) and filtering (σ_m) channels affect Ξ_{ls}. Furthermore, using (31) and (30), the relative importance of these channels (i.e., the robust-control channel and the robust-filtering channel) in determining asset demands can be written as:

$$\Xi_{ly} \equiv \frac{|\alpha_l|}{|\alpha_y|} = \frac{\sigma_m}{\sigma_y\left(r+\lambda\right)} = \frac{K}{r+\lambda}, \tag{42}$$

which shows that given r and λ, the relative importance of the learning-induced hedging demand to the income hedging demand monotonically rises with the robust Kalman gain (K).

4.2. Effects on Expected Consumption Growth

Using the budget constraint and the consumption and portfolio rules obtained above, the dynamics of consumption can be written as:

$$\begin{aligned} dc_t^* = r &\left\{\frac{1}{r+\rho}\left(\mu_t - m_t\right) + \left(\frac{r-\beta}{r}\right)\psi + \frac{1}{2}r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)\left[\frac{\sigma_y}{r+\rho} + \frac{\sigma_m}{\left(r+\rho\right)\left(r+\lambda\right)}\right]^2 + \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}\right\}dt \\ &+ r\left[\alpha^*\sigma_e dB_{e,t} + \frac{\sigma_y}{r+\rho}dB_{y,t} + \frac{\sigma_m}{\left(r+\rho\right)\left(r+\lambda\right)}dB_{m,t}\right], \end{aligned}$$

which means that the expected consumption growth is:

$$\frac{\mathbb{E}_t\left[dc_t^*\right]}{dt} = r\left\{\left(\frac{r-\beta}{r}\right)\psi + \frac{1}{2}r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)\left[\frac{\sigma_y}{r+\rho} + \frac{\sigma_m}{\left(r+\rho\right)\left(r+\lambda\right)}\right]^2 + \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2}\right\}. \tag{43}$$

From (43), we can infer some interesting implications of induced uncertainty for the consumption dynamics. First, given our REU-OU specification, the consumption function obtained from our model allows consumption and wealth to be negative. As argued in Weil (1993), there are several ways to resolve this issue. One is to assume that the household's discount rate (β) is less than the market rate (r) (i.e., the household is more patient). Carroll (2001) argues that households with $\beta \leq r$ will accumulate asset indefinitely, and in the limit, the income stream becomes irrelevant as consumption comes to be financed increasingly out of capital income. Borrowing constraints are unlikely to be of relevance for such households. This point is clear from (43) since expected consumption growth is always positive if $\beta \leq r$. In this case, the household is patient and will manage to accumulate enough wealth to insure against income fluctuations. However, from a general equilibrium perspective in sense of Huggett (1993) and Wang (2003), there exists a unique general equilibrium in which the equilibrium interest rate $r^* < \beta$. (See Online Appendix C for a proof.) In this general equilibrium, (43) becomes:

$$\mathbb{E}_t \left[dc_t^* \right] = 0, \tag{44}$$

which implies that equilibrium consumption is a martingale as predicted by the standard permanent income model.

Another way to rule out negative consumption is to restrict the downward riskiness of the income process such that consumption would not tend to zero or negative after a long sequence of low income shock realizations. In addition, Weil (1993) also argues that we can impose a restriction that the initial level of financial wealth (w_t) is sufficiently large to guarantee the validity of the optimal consumption rule. (If that a household has a high wealth level, then income uncertainty should not matter much.)

It is relatively easier to argue that households have negative wealth since they are allowed to have negative wealth. As is well-known in the literature, how much households would borrow depends on the natural borrowing limit, i.e., the present value of the lowest possible income the household would receive from now on. (Of course, it is possible that the household might face a stronger liquidity constraint than the natural borrowing limit, if access to credit markets is limited.) It is also clear from (43) that the liquidity constraint problem is *less severe* in our model with both parameter and model uncertainty because the additional demand for precautionary saving tilts up consumption profiles and therefore leads to more wealth accumulation, which may reduce the likelihood of being constrained in the future. In other words, the presence of pervasive uncertainty relaxes the importance of liquidity constraints. As argued in Carroll (2001), the precautionary saving demand interacts with the liquidity constraint because the inability to borrow when times are bad provides an additional motive for accumulating assets when times are good, even for impatient households whose discount rate is greater than the interest rate (i.e., $\beta > r$) in general equilibrium. While it is clear that some households are liquidity constrained, and do not behave as described in our paper, the models presented in this paper seem to account for important aspects of reality that have not been explained by the existing consumption-asset allocation models.

5. Quantitative Results

In this section, we provide guidance on pinning down key PU and MU parameters. Then we use the parameterized model to provide quantitative analysis to further help understand the theoretical results and test the model predictions. In addition, we also quantitatively evaluate the welfare losses caused by PU and MU.

5.1. Estimation of PU Parameters

Our key PU parameters are associated with the unknown mean of the income process. To estimate such an income process, we use household-level information in the Panel Study of Income Dynamics (PSID) that contains both cross-sectional and longitudinal information about household income as we need to identify key parameters. For details on the data, please see Online Appendix D.

The income process we estimate is given as follows:

$$
\begin{aligned}
y_{t+1} &= \mu_t + \rho_y y_t + \omega_y \varepsilon_{t+1}, \\
\mu_{t+1} &= \rho_\mu \mu_t + \omega_\mu \epsilon_{t+1},
\end{aligned}
$$

where the mean of income μ_t is stochastic and follows an AR(1) process; ε_{t+1} and ϵ_{t+1} are the iid standard normal innovations to the y and μ processes, respectively. The key parameters we want to estimate are ρ_y, ρ_μ, ω_y, and ω_μ. We follow Blundell et al. (2008) to use both the cross-sectional and time-dimensional variations in household incomes to identify the key parameters. The basic idea is to derive cross-sectional variance and covariance $\text{cov}(\Delta y_t, \Delta y_{t+j})$ $(j = 0, 1, 2, ..., T))$ and choose parameters to match those empirical counterparts in the data.

The estimated values of the persistence coefficients, ρ_y and ρ_μ, are 0.196 and 0.959, respectively. The estimated values of ω_y and ω_μ are 0.247 and 0.118, respectively. Following the literature, we also normalize household income measures as ratios of the mean for that year. Using the estimates, we can recover the corresponding parameter values used in our continuous-time model: $\rho = 1.314$, $\lambda = 0.034$, $\sigma_y = 0.422$, $\sigma_\mu = 0.121$, and $\rho_{y\mu} = 0.861$.

Other parameters are taken from the literature. Following Campbell and Viceira (2002, Chapter 6) and Campbell (2003), we set the parameter values for asset returns and volatility, and consumption as follows: $\pi = 0.05$, $r = 0.025$, $\sigma_e = 0.16$, and $\rho_{ye} = 0.35$. The magnitude of the EIS (ψ) is a key issue in macroeconomics and asset pricing. For example, Vissing-Jorgensen and Attanasio (2003) estimated the IES to be well in excess of one. Campbell (2003), on the other hand, estimates its value to be well below one. Here we choose $\psi = 0.3$, 0.5, and 0.8 for illustrative purposes.[21]

5.2. Calibration of the RB Parameter

To calibrate the robustness parameter (ϑ), the key parameter for MU in our model, we adopt the calibration procedure outlined in Anderson et al. (2003). Specifically, we calibrate ϑ by using the method of detection error probabilities (DEP) that is based on a statistical theory of model

[21] Guvenen (2006) finds that stockholders have a higher EIS (around 1.0) than non-stockholders (around 0.1). Crump et al. (2015) find that the EIS is precisely and robustly estimated to be around 0.8 in the general population using the newly released FRBNY Survey of Consumer Expectations (SCE).

selection. We can then infer what values of ϑ imply reasonable fears of model misspecification for empirically-plausible approximating models. The model detection error probability denoted by q is a measure of how far the distorted model can deviate from the approximating model without being discarded; low values for this probability mean that agents are unwilling to discard many models, implying that the cloud of models surrounding the approximating model is large. In this case, it is easier for the investor to distinguish the two models. (See Online Appendix E for the detailed calibration procedure.) Figure 1 illustrates how DEP (q) varies with the value of ϑ for different values of ψ. We can see from the left upper panel of the figure that the stronger the preference for robustness (higher ϑ), the less the value of q is, holding other parameters constant. For example, when $\gamma = 3$ and $\psi = 0.5$, $q = 26.7\%$ when $\vartheta = 2$, while $q = 14.4\%$ when $\vartheta = 4$. Both values of q are reasonable as argued in Anderson et al. (2003); Maenhout (2004); and Hansen and Sargent (2007, Chapter 9). We set the value of ϑ to be in the range of $[0, 4]$ in our subsequent quantitative analysis. In addition, we can also see from this figure that given the same value of ϑ, q increases with the value of ψ, and ψ has significant impact on the relationship between ϑ and q. For example, holding other parameter values fixed, if $\vartheta = 4$, q is about 14.4% when $\psi = 0.5$, while it is about 5.6% when $\psi = 0.3$.

5.3. Quantitative Analysis

Using these existing, calibrated, and estimated parameter values, we can now examine how RB affects the steady state conditional variance of the unobservable trend (Σ^*), the volatility of the perceived trend (σ_m), and the robust Kalman gain (K) in the filtering problem. The upper panels of Figure 2 shows how the conditional variance (Σ^*) and the volatility of the perceived trend (σ_m) increases with ϑ for different values of the noise-to-signal ratio (σ_y/σ_μ).[22] In addition, for given values of ϑ, Σ^* and σ_m increases with σ_y/σ_μ. Furthermore, the lower and left panel of Figure 2 shows that RB has a non-trivial impact on the Kalman gain (K) and increases the Kalman gain in the filtering problem. The reason behind this result is that RB affects the Kalman gain via the $\vartheta\sigma_y^2$ term in (14) and the value of this term is not small given the estimated value of σ_y. This result is also consistent with the results obtained in Kasa (2006) in a linear-quadratic problem.

Next, using the theoretical results obtained in the previous subsection, we can also quantitatively explore the importance of different mechanisms in determining precautionary saving. To help understand the results, we present three main findings using Figure 3. First, the upper left panel shows that the total contribution of model uncertainty and parameter uncertainty in precautionary saving is increasing with the robustness parameter ϑ and is robust to different values of the EIS parameter ψ. In particular, when $\vartheta = 2$ and $\psi = 0.3$ (which corresponds to a DEP level of 0.2), MU and PU together account for about 75% of total precautionary saving. Second, as the

[22]In the previous subsection, we estimate $\sigma_y = 0.4218$ and $\sigma_\mu = 0.1207$, which means that the noise-to-signal ratio is 3.52.

upper right panel shows, all the precautionary saving is driven by MU and the interaction with PU (corresponding to the ratio, $\Gamma_{MU}/\Gamma_{PU,MU}$, greater than 1), while the contribution of *pure* PU in precautionary saving is slightly negative. This may seem a little surprising at the first glance, but it turns out to be reasonable. In the absence of MU, the effect of PU on precautionary saving is actually negative. This can be seen by comparing Γ_0 and $\Gamma_{PU,0}$ in (34) and (35): as $\sigma_u > \sigma_{m,0}$, we have $\Gamma_0 > \Gamma_{PU,0}$. The economic intuition is that when the agent has enough information to tell the true size of the volatility of the growth mean (which is larger than the perceived volatility), he will save more. Third, as the two middle panels show, as the degree of RB increases, the interaction between MU and PU (the robust filtering channel), as captured by $\Gamma_{MU,filtering}$, plays a greater role in creating precautionary saving, while the share of precautionary saving generated only by the robust control channel (captured by $\Gamma_{MU,control}$) declines. This decomposition highlights the importance of the interaction between MU and PU in driving up the demand for precautionary saving. The two lower panels show the size of precautionary saving relative to the level without MU and PU (i.e., Γ_0). They show that the overall effect of MU and PU on precautionary saving increases rapidly with the increase in the degree of robustness.

Figure 4 illustrates how the total amount of financial wealth invested in the risky asset (α) and its three main components vary with the degree of RB given the estimated parameter values of the income processes reported in Section 5.1. As the blue solid line shows, the total risky-asset holding decreases with the degree of model uncertainty due to RB (ϑ). This decline is largely driven by the decline in the traditional speculation demand (α_s), which means more model uncertainty reduces the demand for risky assets. In addition, as the black dashed line shows, the learning-induced hedging demand (α_l) becomes more negative as the degree of RB increases, which suggests that higher model uncertainty also reduces risky-asset demand through the learning channel (or the robust filtering channel). Indeed, as the upper middle panel in Figure 5 shows, the relative importance of the learning-induced hedging demand increases with the degree of RB, highlighting again the importance of the interaction between MU and PU. In contrast, the income hedging demand (α_y) is a small negative number but independent of RB. As Figure 5 shows, these findings are very robust to different parameter values. Overall, these two figures clearly show that the three components in the total demand for risky assets play different roles and vary differently with the degree of RB.

5.4. Testable Implications of Ignorance for Portfolio Choices

The theoretical and quantitative results in the previous sections suggest that more ignorance leads to less holding of risky assets. As one testable implication, we show this is consistent with the data. We test this prediction in two steps. First, there is evidence that highly educated people are usually more financially literate (i.e., less ignorant). For example, Lusardi and Mitchell (2014) provide a recent survey testing participants' command of financial principles and planning by using their three-question poll about compounding, inflation, and risk. They find that people with more education did better. In the U.S., for example, 44.3% of those with college degrees answered all three questions correctly, compared with 31.3% of those with some college, 19.2% of those with only a high school degree, and 12.6% of those with less than a high school degree. Among those with post-graduate degrees, 63.8% got all answers right. Other countries showed similar results. This suggests that people with higher education probably have better knowledge about the structure and parameters of the model, and are thus less ignorant about the model specification and parameter uncertainty.

Second, we show that the education level is positively correlated with risky-asset holdings. To do this, we use data from the Panel Study of Income Dynamics (PSID), which contains information about individual households' wealth, income, and stock holdings. In particular, we divide households into three groups by their educational level, and then examine their holdings of stocks, in both absolute terms and relative terms. The details of the data set are explained in Online Appendix D. We report the number of households in each category in Table 1 and summarize the main findings in Tables 2 and 3. Specifically, Table 2 shows that the amount of stock holdings increase with the educational level, not only for the whole sample, but also by income groups.[23] To further control the effects of education on income that also influences the absolute level of stockholding, we report the relative stockholding, defined as the share of stockholding in households' total wealth, in Table 3. In Table 3, when calculating households' wealth, we consider two cases: one includes home equities and one excludes home equities. It is clear from the table that the share of wealth invested in stocks is positively correlated with the household's educational level in both cases. These findings can thus be consistent with our model's predictions and highlight the importance of ignorance-induced uncertainty in explaining the data. Specifically, less well-educated investors probably face greater ignorance-induced uncertainty; consequently, they rationally choose to invest less in the stock market even if the correlation between their labor income and equity returns is the same as that for the well-educated investors.[24]

In the above argument, we have cited the evidence in the literature that highly educated people are usually more financial literate or less ignorant. To further examine if our model is consistent with this evidence, we conduct the following calibration exercise. Specifically, we use our model to qualitatively infer the degree of model uncertainty for each educational group and examine whether our model also supports the argument that highly educated people face less model uncertainty or are less ignorant than low educated people. To implement this, we first fix the parameters

[23]This is consistent with earlier work in the literature. For example, Tables 1 and 2 in Luo (2017) show a positive relationship between the mean value of stockholding and the education level at all income and net worth levels using the Survey of Consumer Finances (SCF) data. Haliassos and Bertaut (1995) also find that the share invested in the stock market is substantially larger among those with at least a college degree compared to those with less than a high school education at all income levels.

[24]As documented in Campbell (2006), there is some evidence that households understand their own limitations and constraints, and avoid investment opportunities for which they feel unqualified.

that are not related to ignorance and induced uncertainty at the same values as in the previous subsection: $\pi = 0.05$, $r = 0.025$, $\sigma_e = 0.16$, $\rho_{ye} = 0.35$, $\rho = 1.631$, $\lambda = 0.042$, and $\rho_{y\mu} = 0.897$. Second, we use PSID data to separately estimate the key parameters for parameter uncertainty, i.e., σ_y/σ_μ. The returned values for the low educational group (i.e., Less than High School), the middle educational group (i.e., High School and Some College), and the high educational group (i.e., College and Above) are 5.90, 2.78, and 2.57 respectively.[25] Finally, after fixing all other parameters, we use the relative holding of risky assets to calibrate the parameter for model uncertainty, ϑ, for each group. As we normalize the relative risky asset holding for the low educational group to be 1, we also normalize the value of ϑ for the high educational group (which potentially faces less model uncertainty) to be 0. So, this calibration exercise will tell us if the two lower educational groups do have higher degrees of model uncertainty. The results are reported in Table 4.

There are two main findings in Table 4. First, our calibration results do suggest that more educated households face less amount of model uncertainty. As the middle panel shows, the implied DEP for the lowest educational group is 0.04, which is lower than the 0.16 for the middle group, and further lower than the highest educational group. As a lower DEP means that it is easier to distinguish the distorted model from the approximating model (or the distorted model is more different from the approximating model in a statistical sense), our calibration results suggest that households with less education face more model uncertainty. Second, as shown in the bottom panel, the full-information rational expectation model (FI-RE) cannot explain the differences in risky asset holding across different educational groups, suggesting that incorporating model uncertainty and parameter uncertainty is important to explaining the data.

It is also worth noting that when we shut down the model uncertainty channel, parameter uncertainty itself cannot explain the observed patterns of asset holdings. Holding other parameter values fixed, the model without model uncertainty implies that the two ratios are 1.01 and 1.02, respectively, which are well below the empirical counterparts.

6. The Welfare Costs of Ignorance

We have shown how the interactions of the two types of ignorance alter individual's optimal consumption and portfolio choices. The other important question is whether ignorance leads to significant welfare loss. If so, how large could it be? In our model, compared to the FI-RE case, the investor with incomplete information about income trend makes consumption and portfolio decisions deviating from the first-best path. In other words, having more precise information about income trends can improve the investor's welfare. We measure the welfare cost of ignorance in two ways: the marginal effects and the long-run effects. The basic idea is to measure the amount

[25] In this estimation exercise we apply a slightly stricter sample selection rule (by raising the threshold from 1% to 5% in removing income outliers) in order to make sure the solution for the low education group is an internal solution (i.e., non-negative).

of consumption or income an average investor is willing to give up to reduce such uncertainty due to ignorance.

6.1. Local Welfare Effects

Rather than removing all uncertainty, in this subsection, we follow Barro (2009)'s local welfare analysis and define a cost that measures the welfare benefits from reducing induced uncertainty at the margin. This welfare analysis approach allows us to answer questions such as "how much investors would pay to reduce partial uncertainty, such as 10% of parameter uncertainty, in order to keep the level of lifetime utility unchanged." In addition, the marginal analysis is useful because most economic policies would not be designed to eliminate uncertainty entirely and thus calculating the potential benefits at the margin may be useful in itself. To examine the *marginal* effects of induced uncertainty due to ignorance on welfare, we follow Barro (2009) to compute the marginal welfare costs due to induced uncertainty at different degrees of robustness (ϑ). The basic idea of this calculation is to use the value function (32) to calculate the effects of induced uncertainty on the expected lifetime utility and compare them with those from proportionate changes in the initial level of consumption.

Given the complexity of the value function, (32) and (33), we do the welfare analysis by considering a typical investor in a general equilibrium in which the two constant saving components, the precautionary saving demand and the saving demand due to relative impatience, are exactly canceled out. Specifically, following Huggett (1993), Calvet (2001), and Wang (2003), we assume that the economy is populated by a continuum of *ex ante* identical, but *ex post* heterogeneous agents, of total mass normalized to one, with each agent solving the optimal consumption and savings problem with parameter and model uncertainty proposed in the previous section. (See Online Appendix C for the proof on the existence and uniqueness of the general equilibrium.)

The following proposition provides the result on the welfare costs of ignorance in general equilibrium:

Proposition 3. *The marginal welfare costs* (mwc) *due to model uncertainty for different degrees of parameter uncertainty measured by σ_y/σ_μ can be written as:*

$$\begin{aligned}\text{mwc}\,(\vartheta) &\equiv -\frac{\partial V/\partial\vartheta}{(\partial V/\partial c_0)\,c_0} \\ &= \frac{\psi}{c_0}\left\{\frac{2\beta\psi^2\,(r^*+\rho)^2}{\sigma_y^2 r^{*2}\left[1+\frac{\sigma_m}{(r^*+\lambda)\sigma_y}\right]^2}+\psi\widetilde{\gamma}\left[1-\frac{2r^*}{r^*+\rho}-\frac{2r^*\frac{\sigma_m}{\sigma_y}}{(r^*+\lambda)\left(r^*+\lambda+\frac{\sigma_m}{\sigma_y}\right)}\right]\right\}^{-1},\end{aligned} \tag{45}$$

where r^* is the equilibrium risk-free rate, $\partial V/\partial\vartheta$ and $\partial V/\partial c_0$ are evaluated in general equilibrium for given c_0. We then use $\Omega = \text{mwc}\,(\vartheta)\cdot 0.1\cdot\vartheta$ to measure how much initial consumption should be increased to *make the investor's lifetime utility unchanged when there is* 10% *increase in* ϑ.

Proof. Using the general equilibrium condition:[26]

$$\frac{1}{2}r^{*}\left(\gamma+\frac{\vartheta}{\psi}\right)\frac{\sigma_{y}^{2}}{\left(r^{*}+\rho\right)^{2}}\left[1+\frac{\sigma_{m}}{\left(r^{*}+\lambda\right)\sigma_{y}}\right]^{2}-\psi\left(\frac{\beta}{r^{*}}-1\right)=0,$$

where r^{*} is the equilibrium interest rate, $c_0 = -\psi \ln\left(r^{*}/\beta\right) + J_0$, and $J_0 = -\alpha_0 - \alpha_1 w_0 - \alpha_2 y_0 - \alpha_3 m_0$, the value function, (32), can be rewritten as:

$$V_0 = -\frac{\beta\psi}{r^{*}}\exp\left(-\frac{1}{\psi}c_0\right),$$

where c_0 is consumption at the initial period. The equilibrium interest rate, r^{*}, is determined by:

$$G=\frac{1}{2}r^{*}\widetilde{\gamma}\frac{\sigma_{y}^{2}}{\left(r^{*}+\rho\right)^{2}}\left[1+\frac{\sigma_{m}}{\left(r^{*}+\lambda\right)\sigma_{y}}\right]^{2}-\psi\left(\frac{\beta}{r^{*}}-1\right)=0.$$

We can then easily obtain that:

$$\begin{aligned}
\frac{\partial V_0}{\partial c_0} &= \frac{\beta}{r^{*}}\exp\left(-\frac{c_0}{\psi}\right),\\
\frac{\partial V_0}{\partial \vartheta} &= \frac{\partial V_0}{\partial r^{*}}\frac{\partial r^{*}}{\partial \vartheta}\\
&= -\frac{\beta}{r^{*}}\exp\left(-\frac{c_0}{\psi}\right)\psi\left\{\begin{array}{c}\frac{2\beta\psi^{2}\left(r^{*}+\rho\right)^{2}}{\sigma_{y}^{2}r^{*2}\left[1+\frac{\sigma_{m}}{\left(r^{*}+\lambda\right)\sigma_{y}}\right]^{2}}\\ +\psi\widetilde{\gamma}\left[1-\frac{2r^{*}}{r^{*}+\rho}-\frac{2r^{*}\frac{\sigma_{m}}{\sigma_{y}}}{\left(r^{*}+\lambda\right)\left(r^{*}+\lambda+\frac{\sigma_{m}}{\sigma_{y}}\right)}\right]\end{array}\right\}^{-1},
\end{aligned}$$

where we use the fact $\frac{\partial r^{*}}{\partial \vartheta} = -\frac{\partial G/\partial \vartheta}{\partial G/\partial r^{*}}$. Using these expressions, it is straightforward to compute $\text{mwc}\left(\vartheta\right) = -\frac{\partial V_0/\partial \vartheta}{\left(\partial V_0/\partial c_0\right)c_0}$. ■

The value of $\text{mwc}\left(\vartheta\right)$ provides the proportionate increase in initial consumption that compensates, at the margin, for an increase in the degree of RB (i.e., an increase in model uncertainty) for a given σ_y/σ_μ – in the sense of keeping the level of lifetime utility unchanged. To provide some quantitative results, we use the same values as in the previous section: $\psi = 0.5$, $\sigma_e = 0.16$, $\rho_{ye} = 0.35$, $\rho = 1.631$, $\lambda = 0.042$, $\sigma_y = 0.455$, $\sigma_\mu = 0.121$, and $\rho_{y\mu} = 0.897$, and set set $c_0 = 0.4\overline{y}$.[27] Figure 6 illustrate how the welfare costs due to ignorance (Ω) vary with ϑ for different values of σ_y/σ_μ. It is clear from this figure that the welfare costs of ignorance are nontrivial for plau-

[26]See Online Appendix C for the derivation of this equilibrium condition.

[27]PSID has less information about durables expenditures. For nondurables, depending on whether we use before or after tax income, the ratio of nondurables to gross income are 27% and 38%, respectively. As durables are about half of nondurables, total consumption as a share of income would be 40% or 57%.

sible parameter values. For example, when $\gamma = 3$, $\vartheta = 2$, $\sigma_y/\sigma_\mu = 3.5$, we can calculate that $\text{mwc}(\vartheta) = 0.1$, which suggests that a 10% increase in ϑ (from 2 to 2.2, the corresponding reduction in DEP, p, from 0.245 to 0.229, is about 6.7%) requires an increase in initial consumption by 2% (i.e., $\Omega = \text{mwc}(\vartheta) \cdot 0.1 \cdot \vartheta = 2\%$) to make his lifetime utility unchanged. In other words, a typical investor in our model economy would be willing to sacrifice 2% of his initial consumption to reduce the degree of robustness by 10% (or the amount of model uncertainty, p, by about 6.7%). In this case, if the investor's initial consumption (c_0) is $\$30,000$, he would pay $600 to remove the ignorance due to model uncertainty.

Furthermore, holding other parameter values unchanged, when the degree of parameter uncertainty (σ_y/σ_μ) increases from 3.5 to 4, we can calculate that $\text{mwc}(\vartheta) = 0.112$, which suggests that a 10% increase in ϑ (from 2 to 2.2) requires an increase in initial consumption by 2.23% (i.e., $\Omega = \text{mwc}(\vartheta) \cdot 0.1 \cdot \vartheta = 2.23\%$) to make his lifetime utility unchanged. The above two numerical examples show that a typical investor in our model economy would be willing to sacrifice more of his initial consumption to reduce model uncertainty when facing more parameter uncertainty.

6.2. Total Welfare Costs due to Ignorance-Induced Uncertainty

In this section, we follow Lucas' elimination-of-risk method (see Lucas 1987) to quantify the welfare cost of ignorance in the general equilibrium. It is worth noting that in Lucas (1987), the welfare cost of aggregate fluctuations is expressed in terms of percentage of consumption the representative agent in the *endowment* economy is willing to give up at all dates to switch to the deterministic economy. However, such a welfare measure may not be very informative when the consumers can choose optimal consumption–saving-portfolio plans because the marginal propensity of consumption out of total wealth is now endogenous and affects consumption growth. Here we follow Obstfeld (1994) and use (32) to assess the effects on attained welfare from large changes in ignorance-induced uncertainty. Let c_0 and r^* be values that apply for our benchmark general equilibrium model. Let $\widetilde{c}_0$ and $\widetilde{r}^*$ be values that apply in an alternative situation that delivers the same attained welfare. The following proposition summarizes the result about how ignorance affects the welfare costs in general equilibrium:

Proposition 4. *The welfare costs due to ignorance-induced uncertainty are given by:*

$$\Delta \equiv \frac{c_0 - \widetilde{c}_0}{\widetilde{c}_0} = \frac{\psi}{\widetilde{c}_0} \ln\left(\frac{\widetilde{r}^*}{r^*}\right), \tag{46}$$

where r^* and $\widetilde{r}^*$ are the equilibrium interest rates in the our benchmark model and the alternative model, respectively.

Proof. Using the value function, (32), we have

$$-\frac{\beta\psi}{r^*}\exp\left(-\frac{1}{\psi}c_0\right)=-\frac{\beta\psi}{\widetilde{r}^*}\exp\left(-\frac{1}{\psi}\widetilde{c}_0\right), \tag{47}$$

which leads to (46). ■

To understand how the welfare cost varies with the degrees of parameter and model uncertainty, we note that:

$$\frac{\partial\Delta}{\partial\vartheta}=\frac{\partial\Delta}{\partial r^*}\frac{\partial r^*}{\partial\vartheta}>0,\frac{\partial\Delta}{\partial\left(\sigma_y/\sigma_\mu\right)}=\frac{\partial\Delta}{\partial r^*}\frac{\partial r^*}{\partial\left(\sigma_y/\sigma_\mu\right)}>0;$$

for plausible values, so that higher ignorance-induced uncertainty leads to larger welfare costs.

To provide some quantitative results, we use the same parameter values as in the previous subsection. Figure 7 illustrate how the welfare costs of parameter and model uncertainty due to ignorance (Δ) vary with ϑ for different values of σ_y/σ_μ. It is clear from this figure that the welfare costs of ignorance are significant for plausible parameter values. For example, when $\gamma=3$, $\vartheta=2$, and $\sigma_y/\sigma_\mu=3.5$, we can calculate that $\Delta=22.6\%$. In other words, a typical investor in our model economy would be willing to sacrifice 22.6% of his initial consumption in order to get rid of such model uncertainty. It is worth noting that these values of welfare costs are not directly comparable to those obtained in Lucas (1987) because our welfare calculations are based on removing all induced uncertainty when individual households make optimal consumption-saving-portfolio choices, while Lucas-type calculations are based on removing business cycles fluctuations in an *endowment* representative agent economy.

Furthermore, holding other parameter values unchanged, when the degree of parameter uncertainty (σ_y/σ_μ) increases from 3.5 to 4, we can calculate that $\Delta=25.1\%$, which suggests that a typical investor facing more parameter uncertainty would be willing to sacrifice more of of his initial consumption in order to get rid of such model uncertainty.

7. Conclusion

In this paper, we have study the implications of two types of ignorance-induced uncertainty, parameter and model uncertainty, for consumption-portfolio rules and precautionary savings in a continuous-time recursive utility model with uninsurable labor income. Specifically, we explicitly solve the model to explore how the two types of ignorance-induced uncertainty interact with intertemporal substitution, risk aversion, and the correlation between the risky asset and labor income. We show that they have distinct impacts on households' decisions on consumption, precautionary savings, and strategic asset allocation, and can help the model better explain the data on household portfolios. Finally, our analysis suggests that the welfare gains from reducing the amount of induced uncertainty due to ignorance can be significant.

8. Appendix

8.1. Solving the RB Model with Unknown Income Growth

In our benchmark model, we first guess that $J_t = -\alpha_0 - \alpha_1 w_t - \alpha_2 y_t - \alpha_3 m_t$. The J function at time $t + \Delta t$ can thus be written as:[28]

$$\begin{aligned} J_{t+\Delta t} \equiv J\left(w_{t+\Delta t}, y_{t+\Delta t}, m_{t+\Delta t}\right) &= -\alpha_0 - \alpha_1 w_{t+\Delta t} - \alpha_2 y_{t+\Delta t} - \alpha_3 m_{t+\Delta t} \\ &\approx -\alpha_0 - \left[\alpha_1 w_t + \alpha_1\left(r w_t + y_t - c_t + \alpha_t \pi\right)\Delta t + \alpha_1 \sigma_e \alpha_t \Delta B_{e,t}\right] \\ &- \left[\alpha_2 y_t + \alpha_2\left(m_t - \rho y_t\right)\Delta t + \alpha_2 \rho_{ye} \sigma_y \Delta B_{e,t} + \alpha_2 \sqrt{1-\rho_{ye}^2}\sigma_y \Delta B_{i,t}\right] \\ &- \left[\alpha_3 m_t + \alpha_3 \lambda\left(\overline{\mu} - m_t\right)\Delta t + \alpha_3 \rho_{ye} \sigma_m \Delta B_{e,t} + \alpha_3 \sqrt{1-\rho_{ye}^2}\sigma_m \Delta B_{i,t}\right]. \end{aligned}$$

Using the above expression for $J_{t+\Delta t}$ and assume that the time interval Δt goes to infinitesimal dt, we can compute the certainty equivalent of J_{t+dt} as follows:

$$\begin{aligned} &\exp\left(-\gamma \mathbb{CE}_t\right) = \mathbb{E}_t\left[\exp\left(-\gamma J_{t+\Delta t}\right)\right] \\ &= \exp\left(-\gamma \mathbb{E}_t\left[-\alpha_1 w_{t+dt} - \alpha_2 y_{t+dt} - \alpha_3 m_{t+dt}\right] + \frac{1}{2}\gamma^2 \operatorname{var}_t\left[-\alpha_1 w_{t+dt} - \alpha_2 y_{t+dt} - \alpha_3 m_{t+dt}\right] + \gamma \alpha_0\right) \\ &= \exp\left(\gamma \alpha_0 - \gamma\left(\partial J\right)^T \cdot \left(s_t + \mathbb{E}_t\left[ds_t\right]\right) + \frac{\gamma^2}{2}\left[\left(\partial J\right)^T \cdot \Sigma \cdot \partial J\right] dt\right) \\ &= \exp\left(-\gamma J_t\right)\exp\left(-\gamma\left(\partial J\right)^T \cdot \mathbb{E}_t\left[ds_t\right] + \frac{\gamma^2}{2}\left[\left(\partial J\right)^T \cdot \Sigma \cdot \partial J\right] dt\right), \end{aligned} \tag{48}$$

where $s_t = \begin{bmatrix} w_t & y_t & m_t \end{bmatrix}^T$, $ds_t = \begin{bmatrix} dw_t & dy_t & dm_t \end{bmatrix}^T$, $\partial J = \begin{bmatrix} J_w & J_y & J_m \end{bmatrix}^T$, and

$$\Sigma = \begin{bmatrix} \alpha_t^2 \sigma_e^2 & \rho_{ye}\sigma_y \alpha_t \sigma_e & \rho_{ye}\alpha_t \sigma_e \sigma_m \\ \rho_{ye}\sigma_y \alpha_t \sigma_e & \sigma_y^2 & \sigma_y \sigma_m \\ \rho_{ye}\alpha_t \sigma_e \sigma_m & \sigma_y \sigma_m & \sigma_m^2 \end{bmatrix}.$$

Solving this equation yields:

$$\mathbb{CE}_t\left[J_{t+dt}\right] = J_t + \left(\left(\partial J\right)^T \cdot \mathbb{E}_t\left[ds_t\right] - \frac{\gamma}{2}\left[\left(\partial J\right)^T \cdot \Sigma \cdot \partial J\right] dt\right). \tag{49}$$

Substituting the expression of $\mathbb{CE}_t$ into the HJB yields:

$$\beta V\left(J_t\right) = \max_{\{c_t, \alpha_t\}} \min_{v_t} \left\{\beta V\left(c_t\right) + \mathcal{D}V\left(s_t\right) + \frac{1}{2\vartheta_t}\left(v_t^T \cdot \Sigma \cdot v_t\right)\right\}, \tag{50}$$

[28]Here $\Delta B_t = \sqrt{\Delta t}\epsilon$ and ϵ is a standard normal distributed variable.

where

$$\mathcal{D}V(J_t) = V'(J_t)\left((\partial J)^T \cdot \mathbb{E}_t[ds_t] + v_t^T \cdot \Sigma \cdot \partial J - \frac{\gamma}{2}\left[(\partial J)^T \cdot \Sigma \cdot \partial J\right]\right).$$

Solving first for the infimization part of the robust HJB equation yields:

$$v_t = -\vartheta_t V'(J_t)\,\partial J \tag{51}$$

Substituting this optimal distortion into the robust HJB equation yields:

$$\beta V(J_t) = \max_{\{c_t,\alpha_t\}}\left\{\beta V(c_t) + V'(J_t)\left((\partial J)^T \cdot \mathbb{E}_t[ds_t] - \frac{\gamma}{2}\left[(\partial J)^T \cdot \Sigma \cdot \partial J\right] - \frac{\vartheta_t}{2}V'(J_t)\left[(\partial J)^T \cdot \Sigma \cdot \partial J\right]\right)\right\}.$$

As in our benchmark model, we assume that $\vartheta_t = -\frac{\vartheta}{V(J_t)}$. Given that $V(J_t) = (-\psi)\exp(-J_t/\psi)$ and $V'(J_t) = \exp(-J_t/\psi)$, the HJB equation reduces to:

$$\beta V(J_t) = \max_{\{c_t,\alpha_t\}}\left\{\beta V(c_t) + V'(J_t)\left((\partial J)^T \cdot \mathbb{E}_t[ds_t] - \frac{1}{2}\widetilde{\gamma}\left[(\partial J)^T \cdot \Sigma \cdot \partial J\right]\right)\right\},$$

where $\widetilde{\gamma} = \gamma + \vartheta/\psi$. The FOCs are:

$$c_t = -\psi\ln\left(-\frac{\alpha_1}{\beta}\right) + (-\alpha_0 - \alpha_1 w_t - \alpha_2 y_t - \alpha_3 m_t), \tag{52}$$

and

$$\alpha_t = -\frac{\pi}{\widetilde{\gamma}\alpha_1\sigma_e^2} - \frac{\sigma_e\alpha_2\rho_{ye}\sigma_y}{\alpha_1\sigma_e^2} - \frac{\sigma_e\alpha_3\rho_{ye}\sigma_m}{\alpha_1\sigma_e^2}. \tag{53}$$

Substituting these expression back to the HJB yields:

$$(-\alpha_1 - \beta)\psi = \left\{\begin{array}{c}\left[\begin{array}{c}-\alpha_1\left\{rw_t + y_t - \left[-\psi\ln\left(-\frac{\alpha_1}{\beta}\right) + (-\alpha_0 - \alpha_1 w_t - \alpha_2 y_t - \alpha_3 m_t)\right] + \alpha_t\pi\right\}\\ -\alpha_2(m_t - \rho y_t) - \alpha_3\lambda(\overline{\mu} - m_t)\end{array}\right]\\ -\frac{1}{2}\widetilde{\gamma}\left(\alpha_1^2\alpha_t^2\sigma_e^2 + \alpha_2^2\sigma_y^2 + \alpha_3^2\sigma_m^2 + 2\alpha_1\sigma_e\alpha_t\alpha_2\rho_{ye}\sigma_y + 2\alpha_1\sigma_e\alpha_t\alpha_3\rho_{ye}\sigma_m + 2\alpha_2\alpha_3\sigma_y\sigma_m\right)\end{array}\right\}.$$

Matching the coefficients, we obtain:

$$\begin{aligned}\alpha_1 &= -r, \alpha_2 = -\frac{r}{r+\rho}, \alpha_3 = -\frac{r}{(r+\rho)(r+\lambda)},\\ \alpha_0 &= \left(1 - \frac{\beta}{r}\right)\psi - \psi\ln\left(\frac{r}{\beta}\right) - \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2} + \frac{\pi\rho_{ye}}{\sigma_e}\left[\frac{\sigma_y}{(r+\rho)} + \frac{\sigma_m}{(r+\rho)(r+\lambda)}\right]\\ &\quad - \frac{\lambda\overline{\mu}}{(r+\rho)(r+\lambda)} + \frac{1}{2}r\widetilde{\gamma}\left(1 - \rho_{ye}^2\right)\left[\frac{\sigma_y}{r+\rho} + \frac{\sigma_m}{(r+\rho)(r+\lambda)}\right]^2.\end{aligned}$$

Substituting these coefficients back to the FOCs, (52) and (53), yields the optimal consumption and portfolio rules under parameter and model uncertainty in the main text. Substituting the value function into $v_t = -\vartheta_t V'(J_t)\,\partial J$ yields:

$$
\begin{aligned}
v_t &= \frac{\vartheta}{(-\psi)\exp(-J_t/\psi)}\exp(-J_t/\psi)\begin{bmatrix} J_w & J_y & J_m \end{bmatrix}^T = \frac{\vartheta}{\psi}\begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix}^T \\
&= -\frac{\vartheta}{\psi}\begin{bmatrix} r & \frac{r}{r+\rho} & \frac{r}{(r+\rho)(\lambda+r)} \end{bmatrix}^T
\end{aligned}
$$

where we use the fact that $\partial J = \begin{bmatrix} J_w & J_y & J_m \end{bmatrix}^T$ and $J_t = -\alpha_0 - \alpha_1 w_t - \alpha_2 y_t - \alpha_3 m_t$.

8.2. Deriving the Loss Function Due to Incomplete Information

Following Sims (Section 5, 2003), we first define the expected loss function due to incomplete information as follows:

$$
L_t = \mathbb{E}_t\left[\widetilde{V}(w_t, y_t, \mu_t) - V(w_t, y_t, m_t)\right], \tag{54}
$$

where $V(w_t, y_t, m_t)$ is the value function obtained from our benchmark model and is given in (32), and $\widetilde{V}(w_t, y_t, \mu_t)$ is the corresponding value function when the income trend is observable, i.e., $m_t = \mu_t$. It is straightforward to show that:

$$
\begin{aligned}
L_t &= \mathbb{E}_t\left[\widetilde{V}(w_t, y_t, \mu_t) - V(w_t, y_t, m_t)\right] \\
&= V(w_t, y_t, m_t)\,\mathbb{E}_t\left[\exp\left(\frac{1}{\psi}\left((\widetilde{\alpha}_0 - \alpha_0) - \frac{r}{(r+\rho)(r+\lambda)}(\mu_t - m_t)\right)\right) - 1\right] \\
&= V(w_t, y_t, m_t)\,\mathbb{E}_t\left[\exp\left(\frac{1}{\psi}(\widetilde{\alpha}_0 - \alpha_0)\right)\exp\left(-\frac{r}{\psi(r+\rho)(r+\lambda)}(\mu_t - m_t)\right) - 1\right] \\
&\simeq V(w_t, y_t, m_t)\,\mathbb{E}_t\left[\exp\left(\frac{1}{\psi}(\widetilde{\alpha}_0 - \alpha_0)\right)\begin{pmatrix} 1 - \frac{r}{\psi(r+\rho)(r+\lambda)}(\mu_t - m_t) \\ +\frac{1}{2}\left(\frac{r}{\psi(r+\rho)(r+\lambda)}\right)^2(\mu_t - m_t)^2 \\ -\frac{1}{6}\left(\frac{r}{\psi(r+\rho)(r+\lambda)}\right)^3(\mu_t - m_t)^3 \\ +\frac{1}{24}\left(\frac{r}{\psi(r+\rho)(r+\lambda)}\right)^4(\mu_t - m_t)^4 + \cdots \end{pmatrix} - 1\right]
\end{aligned}
$$

where α_0 is given by (33) and $\widetilde{\alpha}_0$ is obtained by replacing σ_m in (33) by σ_μ. For Gaussian variables, μ_t and m_t, the higher-order moments (the third and fourth moments) of the loss function can be written as:

$$
\begin{aligned}
&-\frac{1}{6}\left(\frac{r}{\psi(r+\rho)(r+\lambda)}\right)^3(\mu_t - m_t)^3 + \frac{1}{24}\left(\frac{r}{\psi(r+\rho)(r+\lambda)}\right)^4(\mu_t - m_t)^4 \\
&= \frac{1}{8}\left(\frac{r}{\psi(r+\rho)(r+\lambda)}\right)^4\Sigma^2,
\end{aligned}
$$

where we use the facts that for Gaussian variables, we have:

$$\mathbb{E}_t\left[(\mu_t - m_t)^j\right] = \begin{cases} \sigma^j (j-1)!!, \text{ when } j \text{ is even,} \\ 0, \text{ when } j \text{ is odd.} \end{cases},$$

where $m_t = \mathbb{E}_t[\mu_t]$ and σ is the standard deviation of μ. It is straightforward to calculate that the ratio of the sum of the third and fourth moments to the second moment is:

$$ratio = \frac{\frac{1}{8}\left(\frac{r}{\psi(r+\rho)(r+\lambda)}\right)^4 \Sigma^2}{\frac{1}{2}\left(\frac{r}{\psi(r+\rho)(r+\lambda)}\right)^2 \Sigma} = \frac{1}{4}\left(\frac{r}{\psi(r+\rho)(r+\lambda)}\right)^2 \Sigma = 3.48 \times 10^{-3},$$

where we use the fact that $\Sigma = 0.0451$, $\psi = 0.5$, $r = 0.02$, $\rho = 1.314$, and $\lambda = 0.034$.[29] Therefore, minimizing L_t is equivalent to:

$$\min \frac{1}{2} V(w_t, y_t, m_t) \exp\left(\frac{1}{\psi}(\tilde{\alpha}_0 - \alpha_0)\right)\left(\frac{r}{\psi(r+\rho)(r+\lambda)}\right)^2 \mathbb{E}_t\left[(\mu_t - m_t)^2\right]$$
$$\Longleftrightarrow \min \mathbb{E}_t\left[(\mu_t - m_t)^2\right],$$

i.e., the quadratic approximation is accurate and it is just the loss function we used in our robust filtering problem proposed in Section 3.3.

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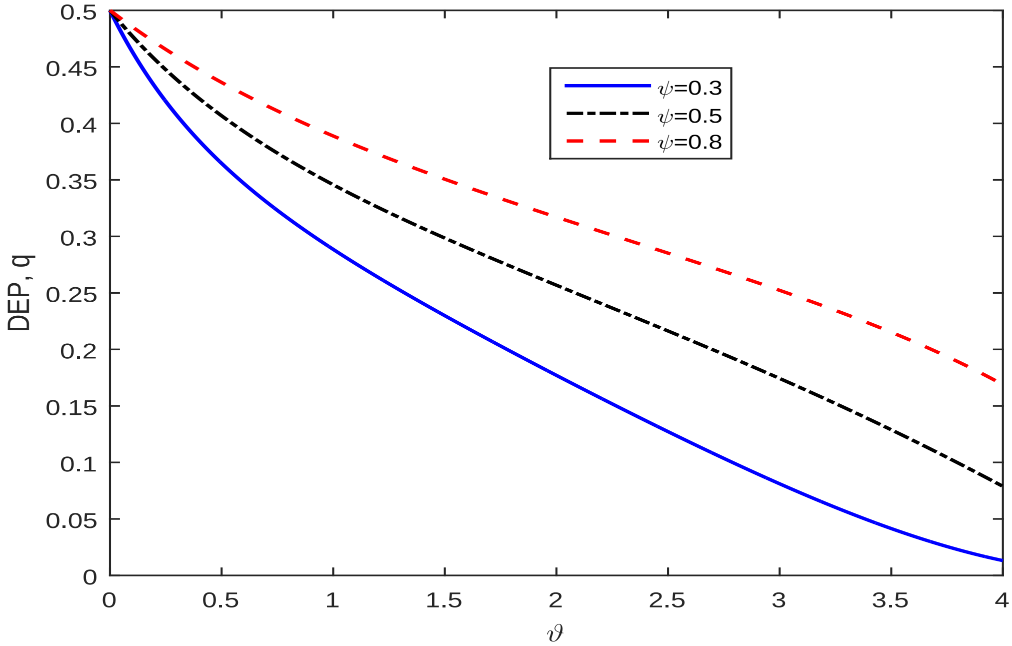

Figure 1. *The Relationship between DEP (q) and RB (ϑ)*

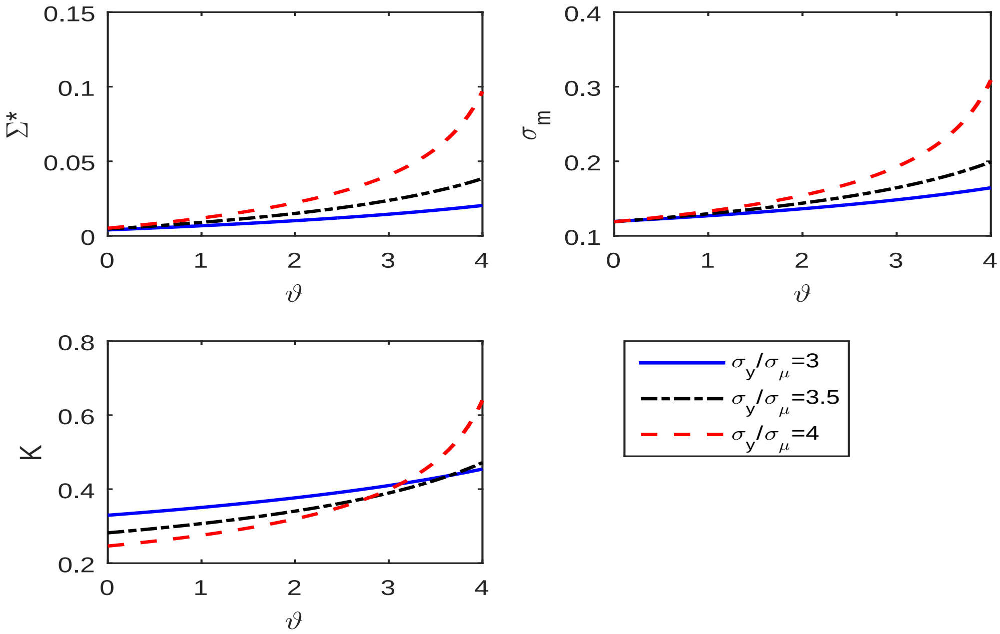

Figure 2. *Effects of RB on the Robust Kalman Gain*

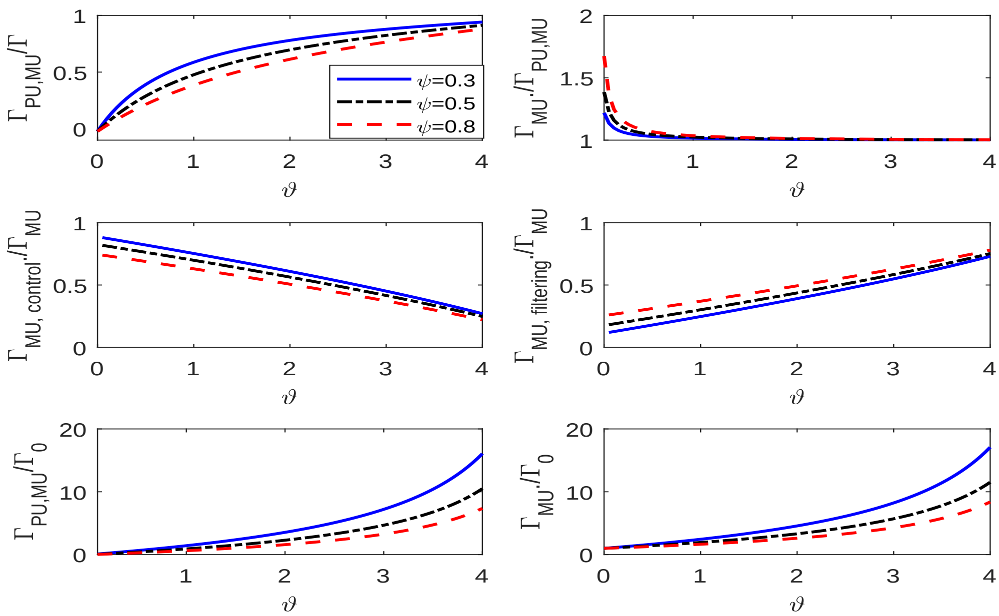

Figure 3. *Decomposition of Precautionary Saving Due to Different Forces*

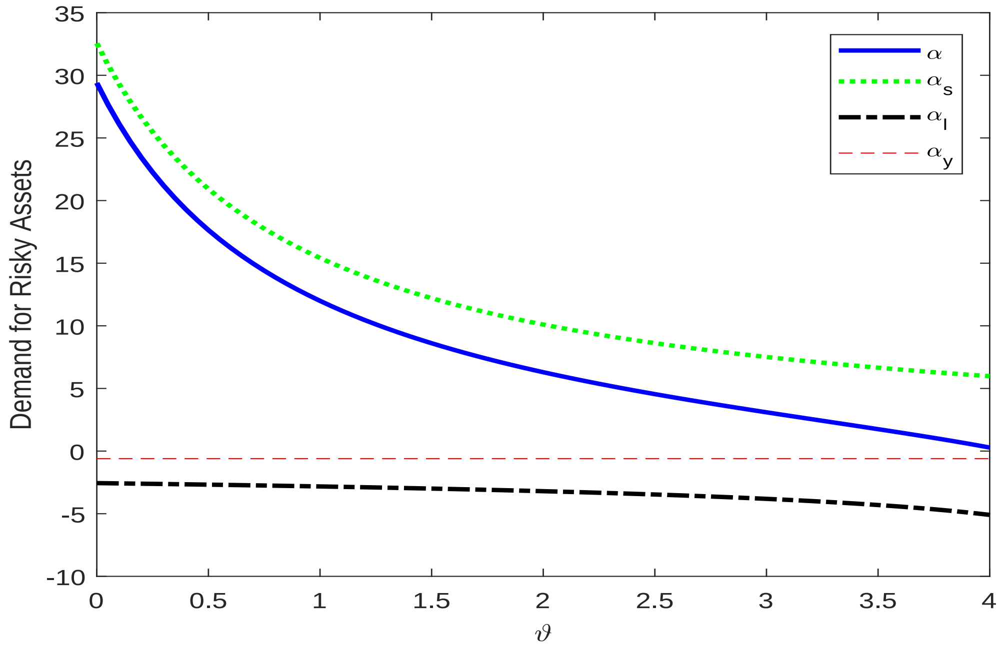

Figure 4. *Three Components in Total Demand for Risky Assets*

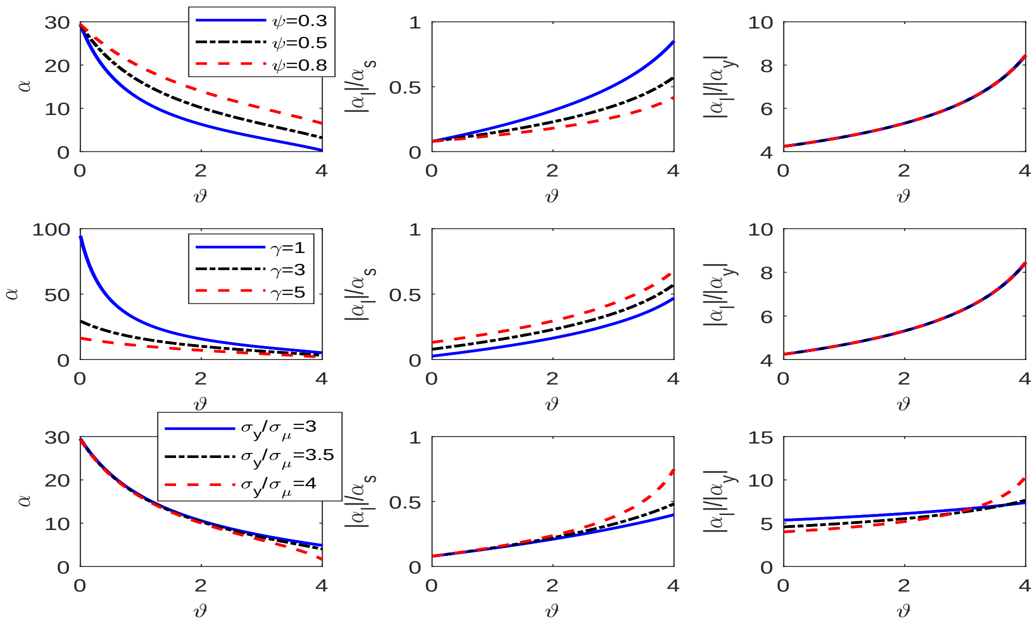

Figure 5. *Effects of Model Uncertainty on Demand for Risky Assets*

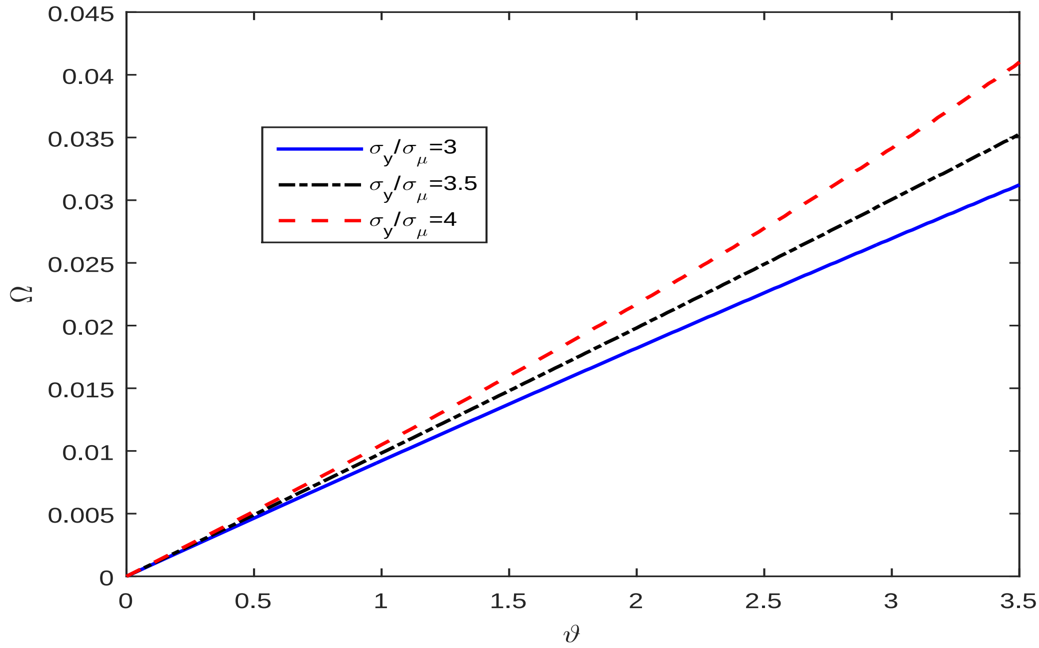

Figure 6. *The Marginal Welfare Costs of Ignorance*

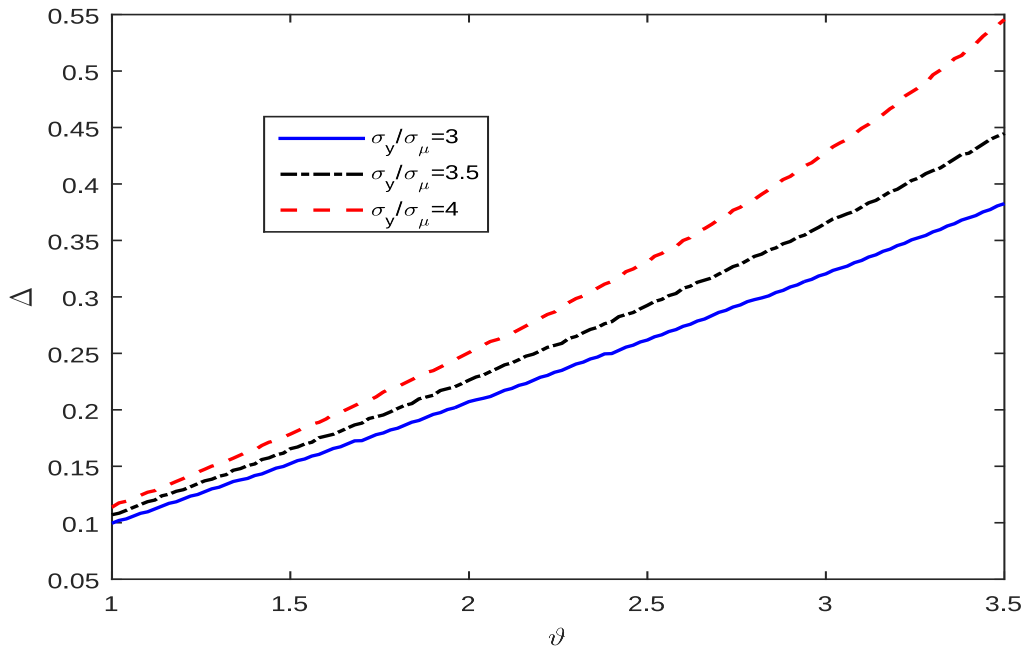

Figure 7. *The Total Welfare Costs of Ignorance*

Table 1. *Number of Households by Education and Income Percentiles*

Income Percentile	Less Than High School	High School and Some College	College and Above
$0-25$	$1,616$	$2,525$	479
$25-50$	728	$2,999$	893
$50-75$	366	$2,809$	$1,445$
$75-100$	163	$1,940$	$2,518$
Full Sample	$2,873$	$10,273$	$5,335$

Table 2. *Mean Stock Values (in Dollars) by Education and Income Percentiles*

Income Percentile	Less Than High School	High School and Some College	College and Above
$0-25$	$2,601.05$	$4,923.80$	$30,479.56$
$25-50$	$6,108.24$	$12,235.73$	$27,215.71$
$50-75$	$7,268.03$	$10,428.84$	$42,343.12$
$75-100$	$16,550.95$	$31,222.16$	$183,824.90$
Full Sample	$4,875.31$	$13,529.19$	$105,522.10$

Table 3. *Mean Stock Values as a Proportion of Wealth (With and Without Home Equity) by Education and Income Percentiles*

Income Percentile	Less Than High School		High School and Some College		College and Above	
	with h.e.	without h.e	with h.e.	without h.e	with h.e.	without h.e
$0-25$	0.89%	1.63%	2.07%	5.87%	6.29%	11.21%
$25-50$	2.03%	3.35%	4.06%	7.12%	8.60%	14.36%
$50-75$	2.03%	6.40%	4.06%	6.77%	9.41%	14.74%
$75-100$	1.59%	5.75%	5.43%	8.57%	12.06%	18.36%
Full Sample	1.42%	2.97%	3.69%	7.00%	10.25%	16.07%

Table 4. *Calibrating RB Parameter (ϑ) using Risky Asset Holdings (α)*

	$\leq$ High School	High School & Some College	College & Above
Data			
Risky-Asset Holding, α (Data)	1 (normalized)	2.8	21.4
PU Parameter, $\frac{\sigma_y}{\sigma_\mu}$ (Data)	5.90	2.78	2.57
Calibration Results			
MU Parameter, ϑ (Model)	5.5	4.0	0
Corresponding DEP	0.04	0.16	0.50
Models' Performance			
α (Benchmark Model)	1 (normalized)	2.8	21.4
α (FI-RE Model)	1 (normalized)	1	1