Robust Investment Strategies with Two Risky Assets

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Abstract

In reality, investors are uncertain about the dynamics of the risky asset returns (e.g., the expected returns and the correlation between the returns of two risky assets). Consequently, investors make robust investment decisions with special concerns on the expected returns and correlations. In this paper, we propose a hierarchical rule for robust investment between two risky assets: select the relatively safe asset first and then decide how much to invest in the relatively risky asset to hedge the ambiguity embedded in the relatively safe asset. After introducing criteria for relative riskiness and cross-hedging for investors with a constant relative risk averse (CRRA) utility, we find that a typical investor would equally invest in the two risky assets regardless of their correlation when they are indistinguishable from the riskiness perspective. Furthermore, the investor will take a long or short position on the relatively risky asset if it can work as the cross-hedging instrument due to their correlation; otherwise, it will not be traded at all. These results provide a unified explanation for the observed "under-diversification", "home bias", and "portfolio inertia" in financial markets from the cross-hedging point of view.

Keywords: Robust investment, ambiguity, model uncertainty

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1 Introduction

Model-based portfolio choice relies heavily on the estimation of model parameters from the historical data of risky assets such as expected returns, volatility, jump components, and correlations between risk factors. The fitted model is then used to characterize the prices and returns of the risky assets. However, investors may be confronted with ambiguity about the dynamics of prices and returns as well as estimation risk. Ambiguity-averse investors will correspondingly take robust strategies. We investigate the optimal portfolio choice of two risky assets when the expected returns and correlation of the two risky assets are uncertain. We focus on the uncertainty about expected returns and correlation because it is well-known that they are difficult to estimate. In addition, we explore the two-asset case to obtain the explicit solution for optimal investment strategies.

Portfolio choice among risky assets is not only a very realistic setting for fund management but also an important issue from a macroeconomic perspective. For example, most equity funds are required to be fully invested in risky assets (Michaud and Michaud, 2008). Many economists also argue that the global economy could be faced with a shortage of safe assets. For example, during the 2007-2009 financial crisis, many of the private safe assets–perceived as safe because they were bestowed with a AAA rating– lost their quality and then disappeared.[1] As a result, the strains associated with the financial crisis quickly lead to concern about the safety of sovereign debts, which leads to a further shrinkage in the global supply of safe assets. It is a generalization of the setting with a risk-free asset in the sense that it can be reduced to the latter case by assigning zero-volatility to one of the risky assets. However, the absence of the risk-free asset may affect the performance of some portfolio rules designed for the investment setting with the risk-free asset. For example, the $1/N$ rule can outperform various sophisticated optimal portfolio rules (DeMiguel, Garlappi, and Uppal, 2009). This shocking result can be turned over when a portfolio rule is designed for a set of risky assets (Kan, Wang, and Zhou, 2020), implying that an optimal portfolio over risky assets is not a trivial generalization of an optimal portfolio with a risk-free asset. From the theoretical point of view, ambiguity is omnipresent within the price dynamics of the risky assets since no one knows the exact evolution of the financial markets. Existing literature has investigated the effect of ambiguity on optimal asset allocation when one cannot accurately estimate one of the model parameters such as the expected return, volatility, or correlation (e.g. Chan, Karceski, and Lakonishok, 1999; Jagannathan and Ma, 2003; Garlappi, Uppal, and Wang, 2007; Epstein and Ji, 2013; Epstein and Halevy, 2019). Note that uncertainty on any one of the model parameters cannot fully capture the ambiguous dynamics of the risky assets. A natural question is what is the robust investment strategy if a fund manager has ambiguity on the driving force of randomness behind the risky assets. We propose a modeling framework to investigate this question, and provide more testable implications for robust investments.

Ambiguity on the price dynamics of risky assets is different from parameter uncertainty due to estimation error. Parameter uncertainty is referred to as the case when an investor knows the true model for the asset price while its parameters cannot be precisely estimated. This setting is formulated with stochastic models defined on a probability space equipped with a unique probability measure, meaning that the investor has complete confidence about the fundamental uncertainty behind the financial market. Aside from parameter uncertainty within such a probability framework, we consider ambiguity on the driving force of randomness behind the price dynamics of risky assets (see

[1] During 2002-2007, the US and European financial markets created large amounts of private safe assets through the securitization of riskier assets.

Hansen and Sargent, 2015; Epstein and Ji, 2021; Luo, Nie, and Wang, 2021, for discussions on the interactions of model uncertainty and parameter uncertainty). In this paper, we characterize an investor's ambiguity on the driving force of market randomness by a set $\mathcal{P}$ of probability measures defined on the canonical space Ω , the set of continuous functions representing the driving force of the market ambiguity. This set $\mathcal{P}$ of probability measures is selected such that the expected returns of risky assets and their correlations are in some intervals, respectively. In our framework, these quantities can be time-varying or random processes with bounded values. This construction method simultaneously accounts for the ambiguity induced by the expected returns and the correlation among risk factors, which is distinctive from the ambiguity induced by one of the risk factors or model parameters (e.g., Chen and Epstein, 2002; Hansen and Sargent, 2008; Liu, 2011; Epstein and Ji, 2013; Luo, 2017; Attaoui, Cao, Duan, and Liu, 2021; Lin and Riedel, 2021).

Portfolio choice can be regarded as selecting some "good" assets among risky assets. The criterion for "good" assets can be up to the preference of investors. Let (μ_1, σ_1) (respectively, (μ_2, σ_2)) be the expected return and volatility of the first (respectively, second) risky asset. We propose a criterion of the relative riskiness is

$$\mathfrak{R}(\mu_1, \sigma_1, \mu_2, \sigma_2) = \frac{\mu_1 - \mu_2}{\sigma_1^2 - \sigma_2^2}, \sigma_1 \neq \sigma_2 .$$

We show that an investor would first choose a relatively safe asset among the risky assets by using such criterion, and then consider if it is necessary to cross-hedge the ambiguity embedded in the relatively safe asset by trading the relatively risky asset. The threshold for cross-hedging is analytically derived for an investor with a CRRA utility, which leads to the conditions for trading. The cross-hedging criterion is

$$\mathfrak{H}(\mu_1, \sigma_1, \mu_2, \sigma_2) = \frac{\sigma_2}{\sigma_1} + \frac{\mu_1 - \mu_2}{\kappa \sigma_1 \sigma_2}, \sigma_1 \neq 0, \sigma_2 \neq 0,$$

where κ governs the degree of risk aversion of an investor in the CRRA utility. This criterion allows an investor to judge whether or not to invest in the relatively risky asset.

In our model, we suppose that the individual takes a stand only on bounds of returns and their correlation of the two risky assets. For any given $\sigma_1 > 0$ and $\sigma_2 > 0$, let σ_1 and σ_2 be the standard deviations (volatility) of two risky assets, respectively. The return process $\mu = (\mu_1, \mu_2)$ of two risky assets takes value in a convex compact set $[\underline{\mu}_1, \overline{\mu}_1] \times [\underline{\mu}_2, \overline{\mu}_2]$ ($\overline{\mu}_i$ and $\underline{\mu}_i$ are constants, and $\overline{\mu}_i > \underline{\mu}_i \geq 0, i = 1, 2$), and their correlation ρ takes value in $[\underline{\rho}, \overline{\rho}]$ ($\underline{\rho}$ and $\overline{\rho}$ are constants, and $-1 \leq \underline{\rho} \leq \overline{\rho} \leq 1$). Using the above criterion, we obtain the following results.[2]

(1) If $\mathfrak{R}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) = \kappa/2$, the risky assets are indistinguishable from the riskiness perspective in the presence of the ambiguity about the expected returns and correlation. An ambiguity-averse investor will invest an equal fraction of her wealth in these two risky assets, regardless of their correlation.

(2) The cross-hedging effect can provide a unified mechanism for explaining "under-diversification", "home bias", and "portfolio inertia", which are widely documented with the empirical evidence.

  (a) In the case that the second risky asset is the relatively safe asset, an investor will buy a proportion of the first risky asset to hedge the ambiguity associated with the second risky asset if $\overline{\rho} < \mathfrak{H}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2)$. Moreover, if $\mathfrak{H}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) < \underline{\rho}$, the investor will short the first risky asset. Otherwise, the investor has no hedging

[2] To the best of our knowledge, these results are new in the literature.

demand for ambiguity embedded in the relatively safe asset. That is, the investor will not trade the relatively risky asset at all, which thus has the potential to generate the phenomenon of limited participation, under-diversification, and home bias we observe in the data. The previous rationale holds in the case that the first risky asset is the relatively safe asset.

(b) The investor may not trade the relatively risky asset for various combinations of the values of the expected return and correlation. This phenomenon is used to be regarded as portfolio inertia, which has been used to account for markets freezing up in response to an increase in uncertainty.

This paper contributes to the literature on robust portfolio choice in three folds. First, we provide a unified mechanism for explaining "under-diversification", "home bias", and "portfolio inertia" from the cross-hedging point of view. In the literature, "under-diversification" is referred to as a bias in individual assets or non-participation in risky assets, "home bias" is the term given to describe the fact that individuals and institutions in most countries hold only modest amounts of foreign equity, and "portfolio inertia" refers to the observation that the list of risky assets or their holdings in the optimal portfolio do not change when the Sharpe ratio of risky assets change. Only when a risky asset can be used to work as a cross-hedging instrument for the existing portfolio, will it be traded by an ambiguity-averse investor. This rationale highlights the effect of both the expected return and correlation between the risky assets in the robust portfolio choice problem. Second, a criterion for the relatively safe asset and a criterion for the cross-hedging demand are proposed to hierarchically construct a robust portfolio. The first stage is to find the relatively safe assets while the second stage is to cross-hedge the ambiguity with the other relatively risky assets. Third, we propose "$1/2$" as a rule-of-thumb for investment between two risky assets when they are indistinguishable from the riskiness perspective for an investor in the worst-case scenario, regardless of their correlations.

Related Literature This paper is related to the literature on robust portfolio choice in absence of a risk-free asset. Robust portfolio choice with parameter uncertainty has been extensively investigated in the existing literature. "Under-diversification", "portfolio inertia", and "home bias" are stylized empirical facts in portfolio research (see, e.g., Cooper and Kaplanis, 1994; Mitton and Vorkink, 2007; Calvet, Campbell, and Sodini, 2007; Van Nieuwerburgh and Veldkamp, 2010; Boyle, Garlappi, Uppal, and Wang, 2012; Guidolin and Liu, 2016). These facts can arise from ambiguity on one of the model parameters such as the expected return or correlation (see, e.g., Uppal and Wang, 2003; Illeditsch, 2011; Pham, Wei, and Zhou, 2021; Jiang, Liu, Tian, and Zeng, 2020; Lin, Sun, and Zhou, 2020; Illeditsch, Ganguli, and Condie, 2021). In a static framework for a correlation ambiguity, Jiang, Liu, Tian, and Zeng (2020) show that an ambiguity-averse investor will exclude one of each pair of assets with a significant point estimation of correlation when they have similar risk-return characteristics. Pham, Wei, and Zhou (2021) provide a justification for under-diversification due to ambiguity on expected returns and correlation in a mean-variance framework. Our continuous-time framework allows us to investigate a fundamental rationale behind these stylized facts from the hedging demand point of view and hierarchical method for portfolio selection.

Portfolio choice in the absence of a risk-free asset affects the performance of portfolio rule (see, e.g., Chiu and Zhou, 2011; Zeng, Li, Li, and Yao, 2016; Lam, Xu, and Yin, 2019; Kan, Wang, and Zhou, 2020). Taking estimation risk into account, Kan, Wang, and Zhou (2020) propose an optimal combining strategy for one-period portfolio choice, which could outperform the $1/N$ rule. Our results coincide with this statement in the sense that it is optimal for an investor to invest equally in the risky assets for the specific market environment. In the continuous-time mean-variance framework, Chiu and Zhou (2011) and Zeng, Li, Li, and Yao (2016) highlight that the efficient frontiers in continuous-time portfolio without a risk-free asset are different from that in the one-period setting. Lam, Xu, and Yin (2019) show that the optimal allocation without a risk-free asset depends linearly on the current wealth while the case with a risk-free asset turns out to be independent of current wealth. The aforementioned papers on portfolio choice without a risk-free asset all take the mean-variance criterion as the objective of the portfolio optimization problem. In contrast, we work with a CRRA preference and investigate how ambiguity on the expected returns and correlation affects the robust investment strategies.

The remainder of this paper is organized as follows. Section 2 investigates the portfolio choice with two risky assets in the absence of ambiguity, focusing on the effect of the correlation between two risky assets. In Section 3, we introduce the model setup for ambiguous dynamics of the risky assets in terms of the expected returns and correlation. The robust strategy is analyzed in detail, which sheds light on the mechanism of some stylized facts from a new point of view. Section 4 provides quantitative analysis on the robust strategies and discusses the economic implications. Section 5 concludes. Proofs are given in the appendix.

2 Investment without Ambiguity

In this section, we consider optimal investment with two risky assets in the absence of ambiguity. The optimal portfolio would shed some light on the robust portfolio when an ambiguity-averse investor is confronted with the ambiguous dynamics of the risky assets.

The dynamics of price processes of two risky assets follows:

$$\begin{cases} dS_{1,t} = \mu_1 S_{1,t}dt + \sigma_1 S_{1,t}dW_{1,t}, \\ dS_{2,t} = \mu_2 S_{2,t}dt + \sigma_2 S_{2,t}(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}), \end{cases} \tag{1}$$

where $\mu_1, \mu_2, \sigma_1 > \sigma_2 \geq 0$ are constants, $\rho \in [-1, 1]$ is the correlation between these two risky assets, $W = (W_1, W_2)$ is a two dimensional Brownian motions on some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, W_1 and W_2 are independent, and $\mathfrak{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by W up to a given time horizon T.

The investor is assumed to invest a proportion π_t of her wealth on the first risky asset at time t. The wealth dynamics X^π follows:

$$\mathrm{d}X_t^\pi = X_t^\pi\left[\pi_t\mu_1 + (1-\pi_t)\,\mu_2\right]\mathrm{d}t + X_t^\pi\left[\pi_t\sigma_1 + (1-\pi)\rho\sigma_2\right]\mathrm{d}W_{1,t} + (1-\pi_t)\sqrt{1-\rho^2}\sigma_2 X_t^\pi \mathrm{d}W_{2,t},$$

with the initial wealth $X_0^\pi = x_0$. We call π a portfolio strategy if π is adapted to the filtration $\mathfrak{F}$ and $\mathbb{E}\left[\int_0^T |\pi_t|^2 dt\right]$ is finite. The portfolio strategy π is *admissible* if $X_t \geq 0, t \in [0, T]$. We denote by Π_1 the set of admissible portfolio strategies.

The investor maximizes her utility at a fixed investment horizon $[0, T]$,

$$V(x_0) = \sup_{\pi\in\Pi_1} \mathbb{E}\left[u\left(T, X_T^\pi\right)\right],$$

where u is a CRRA utility, i.e.,

$$u(T, x) = \frac{Kx^{1-\kappa}}{1-\kappa}, \tag{2}$$

$K > 0$, $\kappa > 0$, and $\kappa \neq 1$.

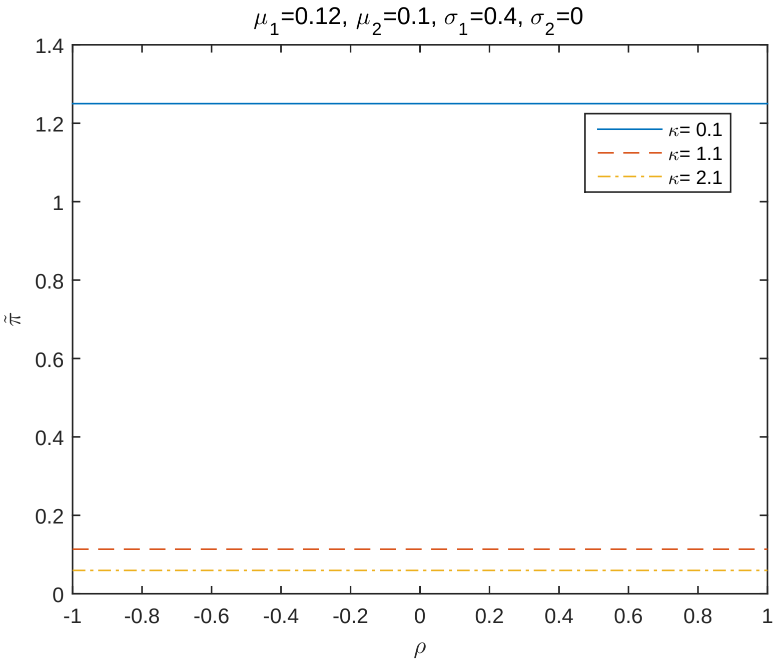


Figure 1: Investment between a risky asset and a risk-free asset

Proposition 2.1. *If an investor with the CRRA utility given by* (2)*, then the optimal investment strategy* $\tilde{\pi}$ *is*

$$\tilde{\pi}_t = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} + \frac{\mu_1 - \mu_2}{\kappa(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)}, \qquad t \in [0, T] \,. \tag{3}$$

Note that if there exists a risk-free asset, the investment strategy can be induced under the assumption that $\sigma_2 = 0$ and the proportion invested in the risky asset is just $(\mu_1 - \mu_2) \,/\, (\kappa\sigma_1^2)$. In this setting, the typical investor invests a fixed fraction of her wealth in the risky asset, as illustrated in Figure 1. Only if the expected return of the risky asset is higher than the risk-free rate, will the investor invest a positive fraction of wealth in the risky asset. For an investor with a much lower degree of risk aversion, she will short the risk-free asset to buy the risky asset ($\tilde{\pi} > 1$). That is, a risk-averse investor will never short the risky asset ($\tilde{\pi} > 0$) unless its expected return is smaller than the risk-free rate. This is not the case if only risky assets are available in the market. The expected return is not the only criterion to assess if the risky asset should be shorted in this setting. For example, an investor may not trade ($\tilde{\pi} = 0$) or even short a risky asset ($\tilde{\pi} < 0$), although its expected return rate is higher than the other one, as illustrated in Figure 2. The investor's preference and the dynamics of the risky assets jointly determine which asset should be shorted. Note that the investment strategy is sensitive to their correlation in the sense that any change in the correlation will lead to an adjustment of the trading position on each risky asset. Any derivation from the correlation coefficient corresponding to $\tilde{\pi} = 0$ drives the investor to optimally change her position away from zero. This will not be the case in the presence of ambiguity about the driving force of the risky assets, as shown in the following sections.

3 Robust Investment Strategies with Ambiguity

In this section, we consider the investment decision with two risky assets when an ambiguity-averse investor is ambiguous about the dynamics of these risky assets in terms of the expected returns and their correlation. For simplicity, we assume that investors have a precise estimate of the variance of the asset returns. This assumption can be justified

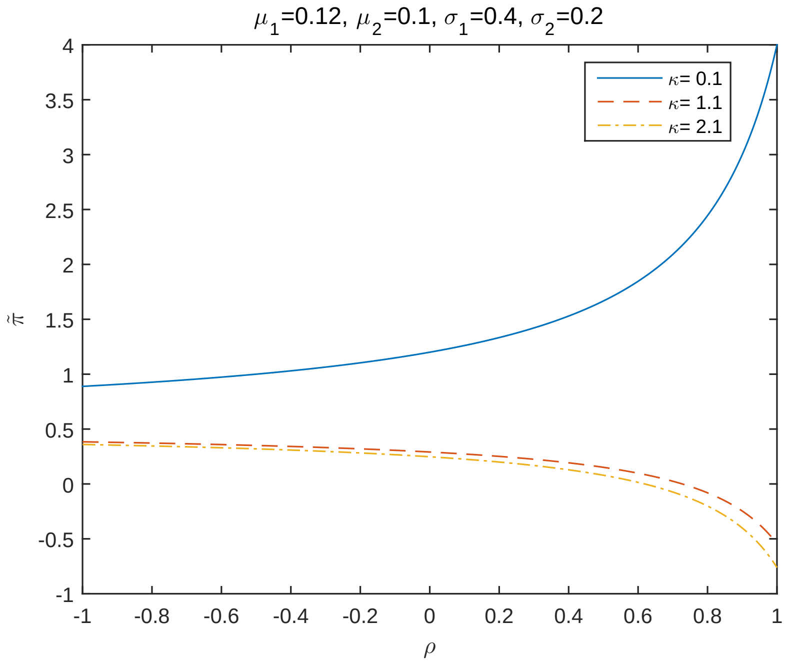


Figure 2: Investment between two risky assets

by both the analytical tractability and the empirical evidence on the predictability of the volatility of asset returns and the difficulties in estimating precisely the expected asset returns and the correlations between them. The investor's ambiguity will be characterized by a set of probability measures on the state space. This formulation allows the expected returns and their correlation to be general time-varying processes rather than constants.

3.1 Model setup

Our model is based on the model proposed in Epstein and Ji (2013). Specifically, we consider a financial market with two tradable risky assets within a fixed investment horizon $[0, T]$. Let $C([0, T], \mathbb{R}^{2+})$ be the set of all continuous paths with positive values in $\mathbb{R}^2$ over the finite time horizon $[0, T]$ endowed with the sup norm. Their price processes $S = (S_{1,t}, S_{2,t})_{0 \le t \le T}$ are modeled by the canonical state space Ω, with

$$\Omega = \left\{ \omega = (\omega(t))_{t \in [0,T]} \in C([0, T], \mathbb{R}^{2+}) : \omega(0) = S_0 \right\},$$

where $S_0 = (S_{1,0}, S_{2,0})$ denotes the initial prices of the risky assets, and $S_t(\omega) = \omega_t$. We equip Ω with the uniform norm and the corresponding Borel σ-field $\mathcal{F}$, and denote by $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T])}$ the natural (raw) filtration generated by S.

A probability measure $\mathbb{P}$ is used to capture the randomness of the risky asset price. Given such a probability measure, one actually knows the price distributions or the dynamics of the risky assets. However, the complexity of the financial market confronts individual investors with ambiguity about these probabilistic characteristics. That is, we cannot assign a unique probability measure on Ω to understand the complex financial market. A set of probability measures will be assigned on Ω rather than a unique one. Such a set of probability measures actually characterizes investors' ambiguity.

For any given $\sigma_1 > 0$ and $\sigma_2 > 0$, let σ_1 and σ_2 be the standard deviations (volatility) of two risky assets, respectively. We construct the set of probability measures such that the mean returns $\mu = (\mu_1, \mu_2)$ of the risky assets taking value in a convex compact set $\Lambda \subset \mathbb{R}_+^2$, and their correlation $\rho \in [\underline{\rho}, \overline{\rho}] \subset [-1, 1]$. In particular, for $\overline{\mu}_i > \underline{\mu}_i \geq 0, i = 1, 2$, we define $\Lambda = [\underline{\mu}_1, \overline{\mu}_1] \times [\underline{\mu}_2, \overline{\mu}_2]$, and $\Theta = \Lambda \times [\underline{\rho}, \overline{\rho}]$. We characterize such ambiguity by Γ^Θ, which is defined as

$$\Gamma^\Theta = \left\{ \theta = (\mu_1, \mu_2, \rho) \mid \rho \in [\underline{\rho}, \overline{\rho}] \text{ and } \mu = (\mu_{1,t}, \mu_{2,t})_{t \in [0,T]} \in \Lambda \text{ are } \mathbb{F}\text{-progressively measurable processes} \right\}.$$

For $\theta \in \Gamma^\Theta$, let $(\Omega, \mathbb{F}, \mathbb{P}^\theta)$ be a filtered probability space such that S is the unique solution of the following stochastic differential equations (SDEs):

$$\begin{cases} dS_{1,t} = \mu_{1,t} S_{1,t} dt + \sigma_1 S_{1,t} dW_{1,t}^\theta, \\ dS_{2,t} = \mu_{2,t} S_{2,t} dt + \sigma_2 S_{2,t} (\rho_t dW_{1,t}^\theta + \sqrt{1 - \rho_t^2} dW_{2,t}^\theta), \end{cases} \tag{4}$$

where $W_1^\theta = (W_{1,t}^\theta)_{0 \leq t \leq T}$ and $W_2^\theta = (W_{2,t}^\theta)_{0 \leq t \leq T}$ are two independent Brownian motions defined on $(\Omega, \mathbb{F}, \mathbb{P}^\theta)$. We denote by $\mathcal{P}^\Theta$ the set of probabilities $\{\mathbb{P}^\theta\}_{\theta \in \Gamma^\Theta}$ on $(\Omega, \mathbb{F})$ such that the SDEs (4) has a unique strong solution.

To interpret such methodology, we define another Brownian motion $\hat{W}_2^\theta = \rho W_1^\theta + \sqrt{1 - \rho^2} W_2^\theta$ on $(\Omega, \mathbb{F}, \mathbb{P}^\theta)$. The price dynamics of the two risky assets (4) can thus be rewritten as:

$$\begin{cases} dS_{1,t} = S_{1,t} \left(\mu_{1,t} dt + \sigma_1 dW_{1,t}^\theta \right), \\ dS_{2,t} = S_{2,t} \left(\mu_{2,t} dt + \sigma_2 d\hat{W}_{2,t}^\theta \right), \end{cases} \tag{5}$$

where W_1^θ and $\hat{W}_2^\theta$ are correlated Brownian motions defined on $(\Omega, \mathbb{F}, \mathbb{P}^\theta)$. Such formulation implies that the prices of the risky assets are driven by two correlated sources of randomness, even though the investor is sure about their volatilities. For each risky asset, the investor is also ambiguous about its expected return.

We assume that the typical investor is endowed with some initial wealth x_0 at time 0, and allocates her wealth between the risky assets. For $\theta \in \Gamma^\Theta$, let π_t be the proportion of her wealth invested in the risky asset S_1 at time $t \geq 0$. The investor's wealth process X^π follows:

$$\mathrm{d}X_t^\pi = X_t^\pi \left[\pi_t \mu_{1,t} + (1 - \pi_t) \mu_{2,t} \right] \mathrm{d}t + X_t^\pi [\pi_t \sigma_1 + (1 - \pi_t) \rho_t \sigma_2] \mathrm{d}W_{1,t}^\theta + (1 - \pi_t) \sqrt{1 - \rho_t^2} \sigma_2 X_t^\pi \mathrm{d}W_{2,t}^\theta, \tag{6}$$

under $\mathbb{P}^\theta$, and $X_0^\pi = x_0$. We denote the set of admissible strategies by $\mathcal{A}(x_0)$, which is defined as follows:

$$\mathcal{A}(x_0) = \left\{ \pi \,\middle|\, \pi \text{ is } \mathbb{F}\text{-adapted}, \int_0^T \pi_r^2 dr < \infty, X_t^\pi \geq 0, t \in [0, T], \ \mathbb{P}^\theta\text{-a.s., for all } \mathbb{P}^\theta \in \mathcal{P}^\Theta \right\}.$$

The objective of an ambiguity-averse investor is to maximize the utility, i.e.,

$$V(x_0) = \sup_{\pi \in \mathcal{A}(x_0)} \inf_{\mathbb{P}^\theta \in \mathcal{P}^\Theta} \mathbb{E}^{\mathbb{P}^\theta} \left[u(T, X_T^\pi) \right], \tag{7}$$

subject to the wealth dynamics (6), where $\mathbb{E}^{\mathbb{P}^\theta}$ denotes the expectation under $\mathbb{P}^\theta \in \mathcal{P}^\Theta$, and u is the CRRA utility defined in (2).

3.2 Optimal portfolio strategies

The solution of (7) is the robust investment strategy for an ambiguity-averse investor when only risky assets are available. It sheds light on how the ambiguity on both the expected returns and correlation affects optimal portfolio choice.

Theorem 3.1. *(I) If $\sigma_1^2 - \sigma_2^2 - (2/\kappa)\left(\underline{\mu}_1 - \underline{\mu}_2\right) = 0$, then the optimal portfolio choice is $\hat{\pi} = 1/2$.*

(II) If $\sigma_1^2 - \sigma_2^2 - (2/\kappa)\left(\overline{\mu}_1 - \underline{\mu}_2\right) > 0$, then we have the following results.

(i) If $\overline{\rho} < \dfrac{\sigma_2}{\sigma_1} + \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$, then the optimal portfolio choice is

$$\hat{\pi} = \frac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} + \frac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)}\,, \tag{8}$$

and $\hat{\pi} \in (0, 1/2)$.

(ii) If $\underline{\rho} \le \dfrac{\sigma_2}{\sigma_1} + \dfrac{\overline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$ and $\dfrac{\sigma_2}{\sigma_1} + \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2} \le \overline{\rho}$, then the optimal portfolio choice is $\hat{\pi} = 0$.

(iii) If $\dfrac{\sigma_2}{\sigma_1} + \dfrac{\overline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2} < \underline{\rho}$, then the optimal portfolio choice is

$$\hat{\pi} = \frac{\sigma_2^2 - \underline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2} + \frac{\overline{\mu}_1 - \underline{\mu}_2}{\kappa(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2)}\,, \tag{9}$$

and $\hat{\pi} \in (-\infty, 0)$.

(III) If $\sigma_1^2 - \sigma_2^2 - (2/\kappa)\left(\underline{\mu}_1 - \overline{\mu}_2\right) \le 0$, then we have the following results.

(i) If $\overline{\rho} < \dfrac{\sigma_1}{\sigma_2} - \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$, then the optimal portfolio choice is

$$\hat{\pi} = \frac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} + \frac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)}\,, \tag{10}$$

and $\hat{\pi} \in (1/2, 1)$.

(ii) If $\underline{\rho} \le \dfrac{\sigma_1}{\sigma_2} - \dfrac{\underline{\mu}_1 - \overline{\mu}_2}{\kappa\sigma_1\sigma_2}$ and $\dfrac{\sigma_1}{\sigma_2} - \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2} \le \overline{\rho}$, then the optimal portfolio choice is $\hat{\pi} = 1$.

(iii) If $\dfrac{\sigma_1}{\sigma_2} - \dfrac{\underline{\mu}_1 - \overline{\mu}_2}{\kappa\sigma_1\sigma_2} < \underline{\rho}$, then the optimal portfolio choice is

$$\hat{\pi} = \frac{\sigma_2^2 - \underline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2} + \frac{\underline{\mu}_1 - \overline{\mu}_2}{\kappa(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2)}\,, \tag{11}$$

and $\hat{\pi} \in (1, +\infty)$.

(IV) If $\sigma_1^2 - \sigma_2^2 - (2/\kappa)\left(\underline{\mu}_1 - \underline{\mu}_2\right) > 0$ and $\sigma_1^2 - \sigma_2^2 - (2/\kappa)\left(\overline{\mu}_1 - \underline{\mu}_2\right) \le 0$, then we have the following results.

(i) If $\dfrac{\sigma_2}{\sigma_1} + \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2} \le \overline{\rho}$, then the optimal portfolio choice is $\hat{\pi} = 0$.

(ii) If $\overline{\rho} < \dfrac{\sigma_2}{\sigma_1} + \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$, then the optimal portfolio choice is

$$\hat{\pi} = \frac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} + \frac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)}\,, \tag{12}$$

and $\hat{\pi} \in (0, 1/2)$.

(V) If $\sigma_1^2 - \sigma_2^2 - \dfrac{2}{\kappa}(\underline{\mu}_1 - \underline{\mu}_2) < 0$ and $\sigma_1^2 - \sigma_2^2 - \dfrac{2}{\kappa}(\underline{\mu}_1 - \overline{\mu}_2) > 0$, then we have the following results.

(i) If $\frac{\sigma_1}{\sigma_2} - \frac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2} \le \overline{\rho}$*, then the optimal portfolio choice is* $\hat{\pi} = 1$.

(ii) If $\overline{\rho} < \frac{\sigma_1}{\sigma_2} - \frac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$*, then the optimal portfolio choice is*

$$\hat{\pi} = \frac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} + \frac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)}\,, \tag{13}$$

and $\hat{\pi} \in (1/2, 1)$.

To interpret the results of Theorem 3.1, we define the relative riskiness of one asset to the other one as:

$$\mathfrak{R}(\mu_1, \sigma_1, \mu_2, \sigma_2) = \frac{\mu_1 - \mu_2}{\sigma_1^2 - \sigma_2^2}\,, \sigma_1 \neq \sigma_2, \tag{14}$$

where (μ_i, σ_i) is the return and volatility of the risky asset i. This measure allows an investor to identify the relatively safe asset between two risky assets. Let us assume that the investor takes asset 1 as the first risky asset and asset 2 as the second one. Then, the rule to identify the relatively safe asset can be summarized in Table 1. We note that the asset with a higher volatility may not be relatively risky over the asset with a lower volatility. The relativeness between two risky assets is also related to an investor's preference.

Table 1: Rule for the relatively safe asset

RSA	$\sigma_1 > \sigma_2$	$\sigma_1 < \sigma_2$
S_2	$\mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) < \kappa/2$	$\mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) > \kappa/2$
S_1	$\mathfrak{R}(\underline{\mu}_1, \sigma_1, \overline{\mu}_2, \sigma_2) > \kappa/2$	$\mathfrak{R}(\underline{\mu}_1, \sigma_1, \overline{\mu}_2, \sigma_2) < \kappa/2$
S_2	$\mathfrak{R}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) < \kappa/2 \le \mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2)$	$\mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) \le \kappa/2 < \mathfrak{R}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2)$
S_1	$\mathfrak{R}(\underline{\mu}_1, \sigma_1, \overline{\mu}_2, \sigma_2) < \kappa/2 < \mathfrak{R}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2)$	$\mathfrak{R}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) < \kappa/2 < \mathfrak{R}(\underline{\mu}_1, \sigma_1, \overline{\mu}_2, \sigma_2)$

Note: 1. RSA is short for relatively safe asset in terms of the measure $\mathfrak{R}$ defined in (14).

2. S_1 and S_2 are relatively equal in riskiness if $\mathfrak{R}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) = \kappa/2$.

If $\mathfrak{R}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) = \kappa/2$, the investor will equally invest in the two risky assets, regardless of their correlation and relative value of volatilities. To the best of our knowledge, this is a novel result in investments. It can work as a rule of thumb for robust investment if the investor would not differentiate these two risky assets in terms of the worst-case relative riskiness. Otherwise, the investor will take the relative riskiness and cross-hedging effect into account when making an optimal portfolio decision.

Note that

$$\sigma_1^2 - \sigma_2^2 - (2/\kappa)\left(\overline{\mu}_1 - \underline{\mu}_2\right) > 0 \Leftrightarrow \begin{cases} \mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) < \kappa/2, & \text{if } \sigma_1 > \sigma_2, \\ \mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) > \kappa/2, & \text{if } \sigma_2 > \sigma_1. \end{cases} \tag{15}$$

This means that the condition of (II) in Theorem 3.1 is equivalent to the right hand of (15). Then, asset 2 will be considered as the relatively safe asset if $\mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) < \kappa/2, \sigma_1 > \sigma_2$ or $\mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) > \kappa/2, \sigma_1 < \sigma_2$. In this case, the investor will invest more in the relatively safe asset but with a conservative belief on its mean return. That is $\pi \in (-\infty, 1/2)$, as indicated by (II) in Theorem 3.1. However, the correlation will affect the investor's belief on the expected return (μ_1) of the relatively risky asset. Actually, the investor is to hedge the ambiguity associated with the relatively safe asset by trading the relatively risky asset. The investor will buy a proportion of the relatively risky asset if their correlation is low enough while shorting the relatively risky asset if their correlation is high enough. The investor has no hedging demand in some cases. More specifically, we define the threshold for crossing hedging as:

$$\mathfrak{H}(\mu_1, \sigma_1, \mu_2, \sigma_2) = \frac{\sigma_2}{\sigma_1} + \frac{\mu_1 - \mu_2}{\kappa \sigma_1 \sigma_2}, \tag{16}$$

where (μ_i, σ_i) is the return and volatility of the risky asset i. If $\overline{\rho} < \mathfrak{H}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2)$, the investor will buy a proportion of the relatively risky asset to hedge the ambiguity associated with the relatively safe asset. Moreover, if $\mathfrak{H}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) < \underline{\rho}$, the investor will short the relatively risky asset. Otherwise, the investor has no hedging demand for ambiguity of the relatively safe asset. Overall, the result of (II) in Theorem 3.1 indicates that the investor first identifies the relatively safe asset, and then hedges its ambiguity with a cross-hedging strategy.

Note that (III) in Theorem 3.1 is the counterpart of (II) in the setting that asset 1 is relatively safe over asset 2. The counterpart of (15) is given as follows:

$$\sigma_1^2 - \sigma_2^2 - (2/\kappa)\left(\underline{\mu}_1 - \overline{\mu}_2\right) \le 0 \Leftrightarrow \begin{cases} \mathfrak{R}(\underline{\mu}_1, \sigma_1, \overline{\mu}_2, \sigma_2) < \kappa/2, & \text{if } \sigma_2 > \sigma_1, \\ \mathfrak{R}(\underline{\mu}_1, \sigma_1, \overline{\mu}_2, \sigma_2) > \kappa/2, & \text{if } \sigma_1 > \sigma_2. \end{cases} \tag{17}$$

In this case, the investor will hedge ambiguity with the relatively risky asset. The threshold for cross-hedging is in a similar way, and the remaining analysis is similar to that for the setting in which asset 2 is the relatively safe asset discussed before.

The condition of (IV) in Theorem 3.1

$$\sigma_1^2 - \sigma_2^2 - (2/\kappa)\left(\overline{\mu}_1 - \underline{\mu}_2\right) \le 0 < \sigma_1^2 - \sigma_2^2 - (2/\kappa)\left(\underline{\mu}_1 - \underline{\mu}_2\right)$$

implies that $\mathfrak{R}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) < \kappa/2 \le \mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2)$, $\sigma_1 > \sigma_2$ or $\mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) \le \kappa/2 < \mathfrak{R}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2)$, $\sigma_2 > \sigma_1$. The investor still takes asset 2 as the relatively safe asset, which is surely invested. If their correlation $\overline{\rho} < \mathfrak{H}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2)$, the investor will hedge the ambiguity associated with asset 2 by taking more than half of wealth of the relatively risky asset. Otherwise, the investor has no hedge demand, and fully invests in the relatively safe asset.

The condition of (V) in Theorem 3.1 can be reformulated in a similar way as the above formulation, which corresponds to the setting in which asset 1 is relatively safe over asset 2. If $\overline{\rho} < \mathfrak{H}(\underline{\mu}_2, \sigma_2, \underline{\mu}_1, \sigma_1)$, asset 1 will be invested with more than half of the investor's wealth when asset 2 can be used as a hedging instrument according to the cross-hedging criterion. Otherwise, the investor will invest all her wealth in the relatively safe asset.

Overall, the investor identifies the safe asset in terms of the relative riskiness between the risky assets. The relatively risky asset may be used to hedge the ambiguity associated with the relative safe asset if their correlation is sufficiently low. The testable implication is that an investor's pool of risky assets consists of relatively safe assets as well as some relatively risky assets with low correlations to the relatively safe assets. However, not all of the risky assets will be traded due to the ambiguous dynamics of the risky assets. This general result can be used to explain the observed "under-diversification" or "home bias" in the data from the robust investment point of view. For example, if some risky asset cannot be used to hedge the ambiguity embedded in the relatively safe asset, it will not be traded, leading to the phenomenon of under-diversification. If family shareholders think that the other risky assets cannot be used to hedge their relatively safe asset, they will hold a large proportion of their wealth in a single firm.

Table 2: Parameters for numerical examples

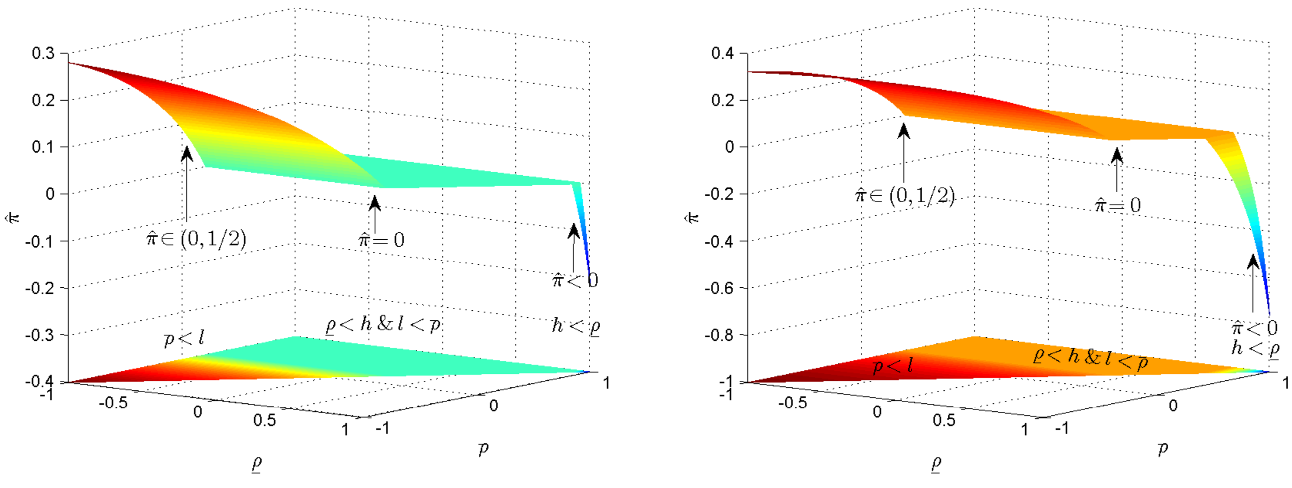
	μ_1	σ_1	μ_2	σ_2
Case 1	$[0.15, 0.25]$	0.35	$[0.20, 0.30]$	0.20
Case 2	$[0.21, 0.30]$	0.35	$[0.10, 0.13]$	0.30
Case 3	$[0.20, 0.32]$	0.35	$[0.25, 0.30]$	0.30
Case 4	$[0.33, 0.40]$	0.35	$[0.20, 0.30]$	0.30
Case 5	0.40	0.35	$[0.20, 0.30]$	0.30
Case 6	0.30	0.35	$[0.25, 0.35]$	0.35

The conservative belief on the expected returns and the correlation between the risky assets is associated with the investor's position. Whenever longing a risky asset, she will take the lower bound of its expected return as the worst-case scenario and vice versa. Whenever a risky asset is shorted in the portfolio, the investor will consider the lower bound of their correlation as the worst-case scenario.

4 Quantitative Analysis

In this section, we quantitatively examine the effects of the ambiguous returns of risky assets on robust investment strategies. Note that the results in Theorem 3.1 are associated with the investor's ambiguity preference that affects the investor's judgement on the relative riskiness between the two risky assets and the cross-hedging role of the relatively risky asset. Given the relative riskiness of the risky assets with criterion (14), an investor may not distinguish them from each other, and invests an equal fraction of her wealth in these two risky assets regardless of their correlation, as indicated by (I) in Theorem 3.1. But other investors may not take the same strategy since they have different preferences. The other results in Theorem 3.1 further imply that such differences in the risk preference may make investors exclude some risky assets out of their portfolios, and invest in the relatively risky asset. Skipping the equal investment in (I) of Theorem 3.1, we focus on the other results in Theorem 3.1. The ambiguous correlation $[\underline{\rho}, \overline{\rho}] \subseteq [-1, 1]$ and the other parameters associated with different cases are given in Table 2.

Using the parameters of Case 1 in Table 2, we obtain that the conservative relative riskiness $\mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) = 0.6061$ and $\mathfrak{R}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) = -0.6061$. All investors with a CRRA utility $(\kappa > 0)$ will never invest an equal fraction of their wealth in these two risky assets. For investors with $\mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) < \kappa/2$ and $\sigma_1 > \sigma_2$, risky asset 2 will be considered as the relatively safe asset, as indicated in Table 1. Figure 3 illustrates the robust investment strategies corresponding to different ambiguous correlation for investors with risk preference $\kappa = 2$ and 4, respectively. We can see from the figure that the proportion invested in the relatively risky asset ranges from $-\infty$ to $1/2$, as indicated by (II) in Theorem 3.1. The relatively risky asset may be not traded at all if $\underline{\rho} < \dfrac{\sigma_2}{\sigma_1} + \dfrac{\overline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$ and $\dfrac{\sigma_2}{\sigma_1} + \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2} < \overline{\rho}$, because the relatively risky asset cannot be used to hedge the ambiguity embedded in the relatively safe asset. However, the relatively risky asset will be shorted to hedge the ambiguity associated with the relatively safe asset if $\dfrac{\sigma_2}{\sigma_1} + \dfrac{\overline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2} < \underline{\rho}$.



(a) $\kappa = 2$ (b) $\kappa = 4$

Figure 3: The case of $\Re(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) < \kappa/2$ and $\sigma_1 > \sigma_2$, where $l = \dfrac{\sigma_2}{\sigma_1} + \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$ and $h = \dfrac{\sigma_2}{\sigma_1} + \dfrac{\overline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$.

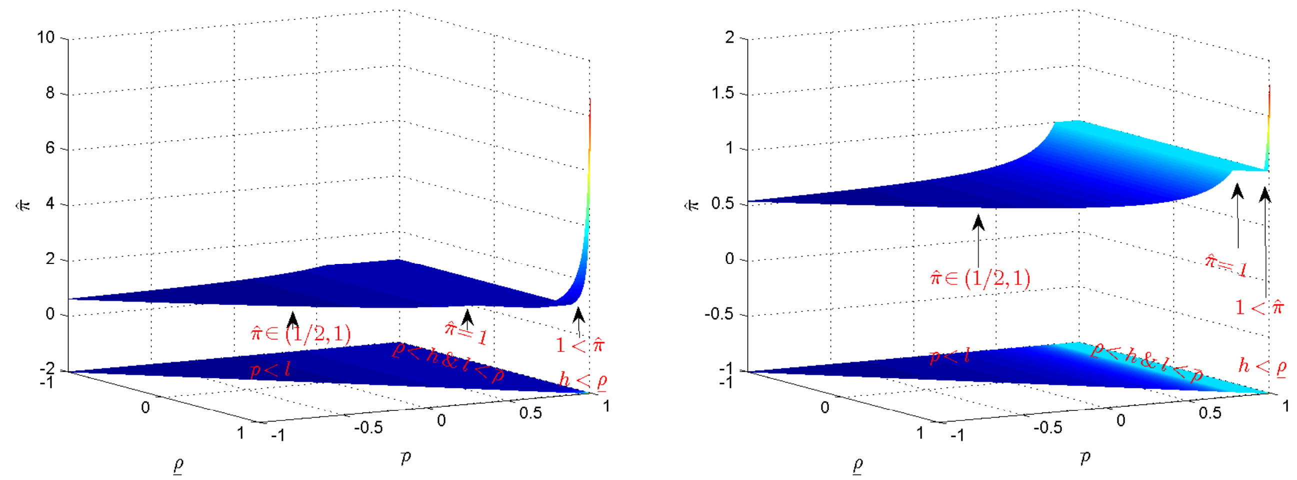
If we adopt the parameter values of Case 2 in Table 2, we obtain the relative riskiness $\Re(\underline{\mu}_1, \sigma_1, \overline{\mu}_2, \sigma_2) = 2.4615$ with $\sigma_1 > \sigma_2$. It is clear that $\Re(\underline{\mu}_1, \sigma_1, \overline{\mu}_2, \sigma_2,) > \kappa/2$ when $\kappa = 2$ or $\kappa = 4$. Recalling the rule for the relatively safe asset in Table 1, we can see that asset 1 is the relatively safe asset. This setting is the counterpart of Case 1 in Table 2. The proportion invested in the relatively safe asset ranges from $1/2$ to ∞, as shown in Figure 4. If $\dfrac{\sigma_1}{\sigma_2} - \dfrac{\underline{\mu}_1 - \overline{\mu}_2}{\kappa\sigma_1\sigma_2} < \underline{\rho}$, the proportion in the relatively safe asset is higher than 1, meaning that the relatively risky asset is shorted in the case. As indicated in Theorem 3.1, the relatively risky asset will not be traded unless it can be used to hedge the ambiguity embedded in the relatively safe asset.

The parameter values of Case 3 in Table 2 correspond to the setting where $\Re(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) < \kappa/2 < \Re(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2)$ for $\kappa = 2$ or $\kappa = 4$. Theorem 3.1 implies that asset 2 will be taken as the relatively safe asset. Asset 1 can work as a cross-hedging instrument if $\overline{\rho} < \dfrac{\sigma_2}{\sigma_1} + \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$. Otherwise, asset 1 will not be traded at all.

The parameter values of Case 4 in Table 2 imply that $\Re(\underline{\mu}_1, \sigma_1, \overline{\mu}_2, \sigma_2) < \kappa/2 < \Re(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2)$ for $\kappa = 2$ or $\kappa = 4$. It is the same setting associated with the last case in Theorem 3.1. In this case, asset 1 is the relatively safe asset, and asset 2 will not be invested unless $\overline{\rho} < \dfrac{\sigma_1}{\sigma_2} - \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$. We can see this pattern in Figure 6, which can be taken as the counterpart of the pattern in Figure 5.

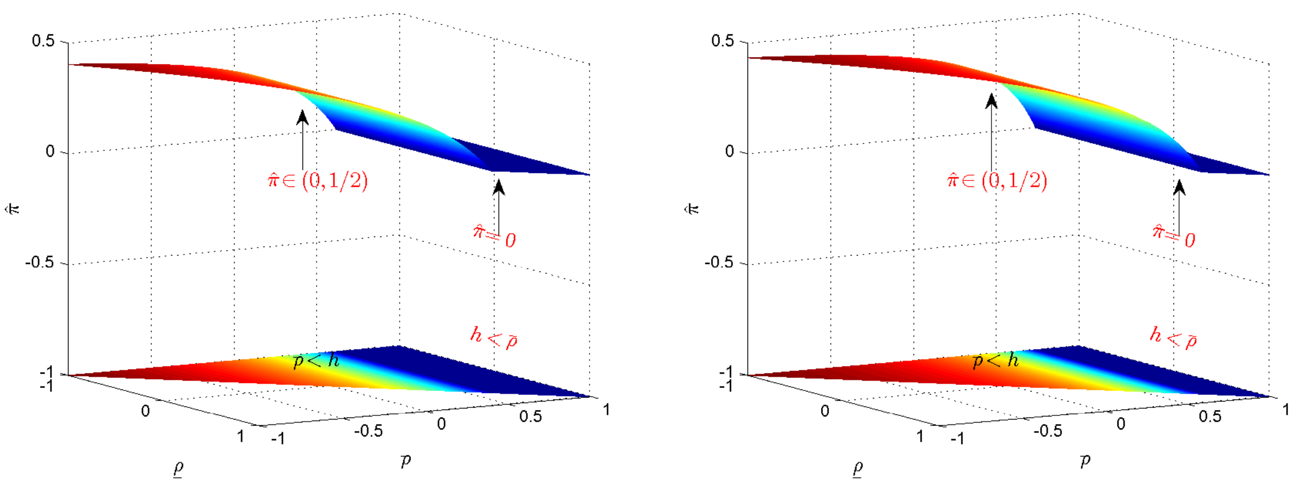
Figures 3-6 also illustrate the portfolio inertia at $\hat{\pi} = 0$ or $\hat{\pi} = 1$. For example, the sub-figure (a) in Figure 3 shows that the investor will not change her position at $\hat{\pi} = 0$ for different bounds of the correlation as long as $l < \overline{\rho}$ and $\underline{\rho} < h$. Furthermore, a small change of position around $\hat{\pi} = 0$ entails a large change in the worst-case belief. That is, the change from the position $\hat{\pi} = -\epsilon$ to $\hat{\pi} = \epsilon$ for an arbitrary small number ϵ ($\epsilon > 0$) is associated with the change of worst-case belief in the correlation from $\underline{\rho}$ to $\overline{\rho}$ as well as the expected return from $\overline{\mu}_1$ to $\underline{\mu}_1$. The other figures indicate the similar patterns with the sub-figure (a) in Figure 3. We omit the detailed analysis of portfolio inertia for these settings.

An application to the home bias puzzle. It is well documented that individuals and institutions in most countries hold only modest amounts of foreign equity. This is puzzling since observed returns on national equity portfolios suggest substantial benefits from international diversification. The home bias in equities was first documented by



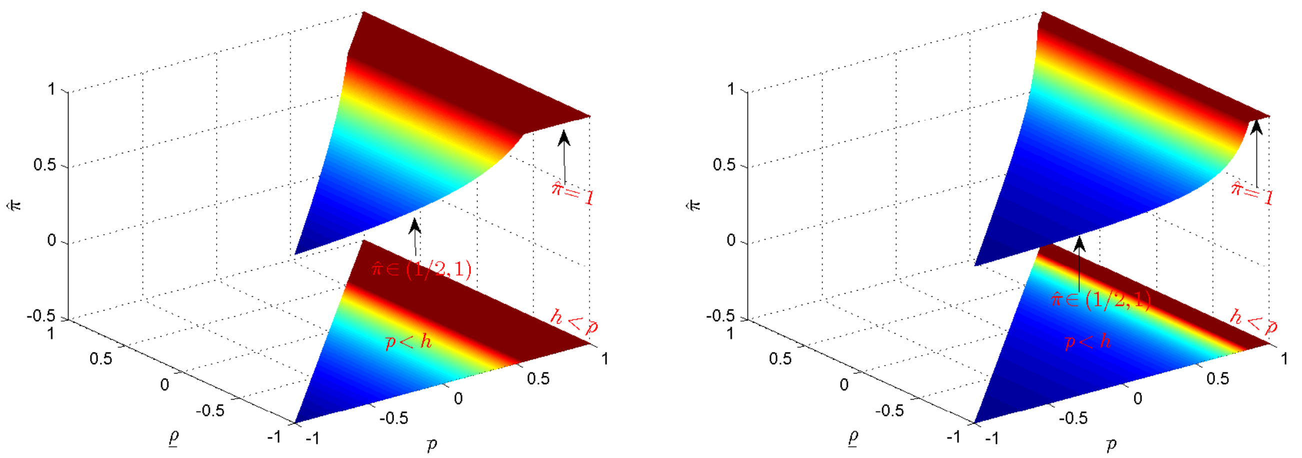
(a) $\kappa = 2$ (b) $\kappa = 4$

Figure 4: The case of $\mathfrak{R}(\underline{\mu}_1, \sigma_1, \overline{\mu}_2, \sigma_2) > \kappa/2$ and $\sigma_1 > \sigma_2$, where $l = \dfrac{\sigma_1}{\sigma_2} - \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$ and $h = \dfrac{\sigma_1}{\sigma_2} - \dfrac{\underline{\mu}_1 - \overline{\mu}_2}{\kappa\sigma_1\sigma_2}$.



(a) $\kappa = 2$ (b) $\kappa = 4$

Figure 5: The case of $\mathfrak{R}(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2) < \kappa/2 < \mathfrak{R}(\overline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2)$, where $h = \dfrac{\sigma_2}{\sigma_1} + \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$.



(a) $\kappa = 2$

(b) $\kappa = 4$

Figure 6: The case of $\Re(\underline{\mu}_1, \sigma_1, \overline{\mu}_2, \sigma_2) < \kappa/2 < \Re(\underline{\mu}_1, \sigma_1, \underline{\mu}_2, \sigma_2)$, where $h = \dfrac{\sigma_1}{\sigma_2} - \dfrac{\underline{\mu}_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$.

French and Poterba (1991) and is called "the home bias puzzle" in the literature. Coval and Moskowitz (1999) find that home bias is not limited to international portfolios by showing that U.S. investment managers exhibit a strong preference for locally headquartered firms.

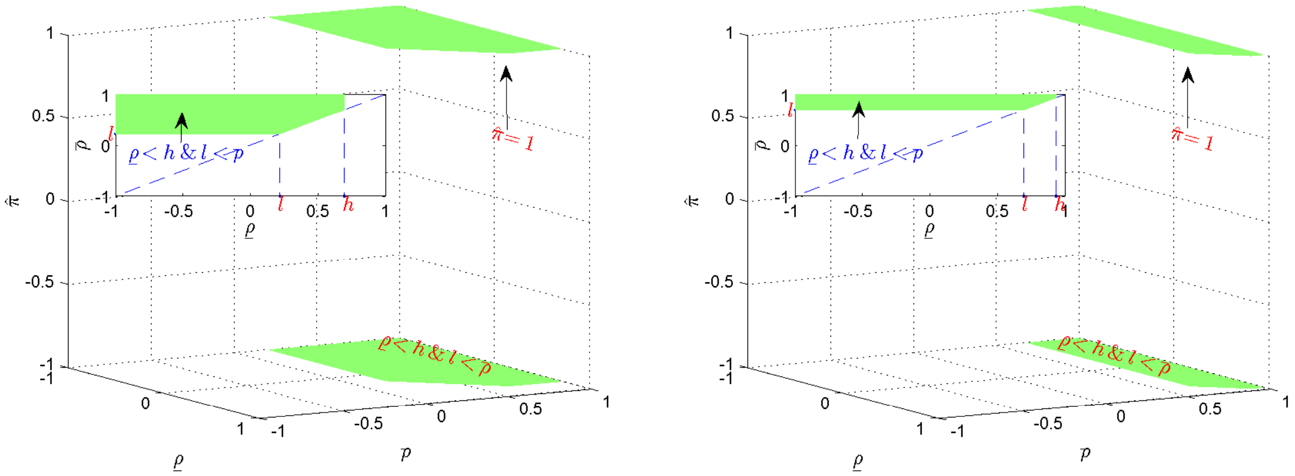
To examine how the ambiguous mean return and correlation affects the phenomenon of the observed home-bias investment, we consider Cases 5 and 6 in Table 2. Simple calculations lead to $\sigma_1^2 - \sigma_2^2 - (2/\kappa)(\mu_1 - \overline{\mu}_2) < 0$ for $\kappa = 2$ or $\kappa = 4$ in Case 5, while $\sigma_1^2 - \sigma_2^2 - (2/\kappa)(\mu_1 - \underline{\mu}_2) < 0$ and $\sigma_1^2 - \sigma_2^2 - (2/\kappa)(\mu_1 - \overline{\mu}_2) > 0$ for $\kappa = 2$ or $\kappa = 4$ in Case 6. These settings are the special cases of (III) and (V) in Theorem 3.1, respectively. Specifically, we assume that the investor is not ambiguous about the dynamics of asset 1, which is referred to as the home asset. In contrast, the investor has some ambiguity on the dynamics of asset 2 (foreign asset) and the correlation between assets 1 and 2. The investor will not trade asset 2 if the ambiguous correlation falls into some special intervals, although its volatility is relatively lower than that of her home asset. For Case 5, the investor invests all her wealth in the home asset (asset 1) in the case of

$$\underline{\rho} < \frac{\sigma_1}{\sigma_2} - \frac{\mu_1 - \overline{\mu}_2}{\kappa\sigma_1\sigma_2} \text{ and } \overline{\rho} > \frac{\sigma_1}{\sigma_2} - \frac{\mu_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2},$$

although its volatility is higher than that of asset 2 (i.e., $\sigma_1 > \sigma_2$). From the cross-hedging point of view, asset 2 is not traded because it cannot be used to cross-hedge the risk associated with the home asset (asset 1). In Case 6, although the two assets has the same volatility, asset 2 is still not traded when the upper bound of their correlation is very high:

$$\overline{\rho} > \frac{\sigma_1}{\sigma_2} - \frac{\mu_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}.$$

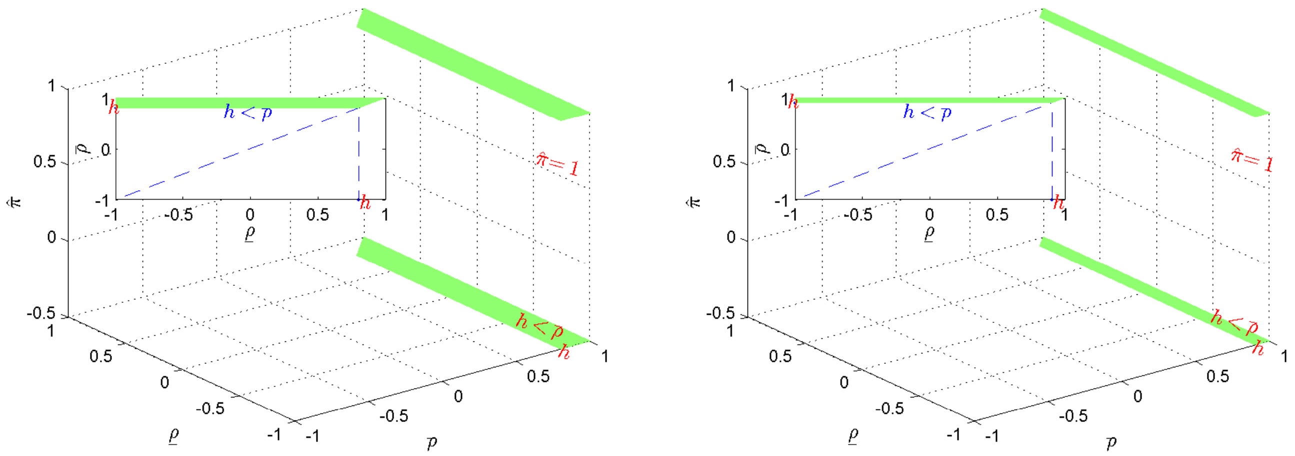
Figures 7 and 8 illustrate these quantitative examples. They clearly show the observed home-bias in portfolio investment from the cross-hedging point of view when the investor is ambiguous about the return of the foreign asset and its correlation with the home asset. As emphasized in the last section that our theoretical model provides a unified mechanism for explaining "under-diversification", "home bias", and "portfolio inertia", the above quantitative exercises can also be used to explain the under-diversification and portfolio inertia observed in the data.



(a) $\kappa = 2$

(b) $\kappa = 4$

Figure 7: Home-bias investment with $\sigma_1 > \sigma_2$: $\sigma_1^2 - \sigma_2^2 - (2/\kappa)\,(\mu_1 - \overline{\mu}_2) \leq 0$, where $l = \dfrac{\sigma_1}{\sigma_2} - \dfrac{\mu_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$ and $h = \dfrac{\sigma_1}{\sigma_2} - \dfrac{\mu_1 - \overline{\mu}_2}{\kappa\sigma_1\sigma_2}$.



(a) $\kappa = 2$

(b) $\kappa = 4$

Figure 8: Home-bias investment with $\sigma_1 = \sigma_2$: $\sigma_1^2 - \sigma_2^2 - (2/\kappa)(\mu_1 - \underline{\mu}_2) < 0$ and $\sigma_1^2 - \sigma_2^2 - (2/\kappa)(\mu_1 - \overline{\mu}_2) > 0$, where $h = \dfrac{\sigma_1}{\sigma_2} - \dfrac{\mu_1 - \underline{\mu}_2}{\kappa\sigma_1\sigma_2}$.

5 Conclusion

Optimal portfolio choice with only risky assets is a realistic investment problem for fund managers or equity investors. In the absence of a risk-free asset, the expected returns and correlation are the key factors that affect the investment strategy. Due to the cognitive limitation or noisy information, ordinary investors may be confronted with ambiguity about the randomness behind the dynamics of these risky assets. In this paper, we have investigated robust portfolio choice between two risky assets, one of which can be a diversified portfolio.

Specifically, we propose a unified framework for modeling the ambiguity about the expected returns and correlation between the risky assets. These quantities can be time-varying or random processes with bounded values. A set of probability measures is assigned on the state space such that the expected returns and their correlation are in some compact sets, respectively. This model setup is reduced to the classical investment problem only if one probability measure lies in this set. That is, the investor has no ambiguity about the dynamics of the risky assets. In this classical setting, we show that one of the risky assets may be shorted although none of the risky assets will be shorted in the presence of a risk-free asset. In the ambiguous setting, the robust portfolio rules are much more complicated.

In this paper, we show that the robust investment problem can be decomposed into a two-stage problem. In the first stage, the typical investor selects a relatively safe asset between the two risky assets according to a criterion, the so-called relative riskiness. In the second stage, the investor will trade the relatively risky asset if it can be used to hedge the ambiguity embedded in the relatively safe asset. Otherwise, the investor will not long or short this asset. The threshold for cross-hedging is analytically obtained to justify if it is necessary to trade the relatively risky asset. This investment rule provides a deep insight on “under-diversification” or “home bias” from the cross-hedging point of view. Moreover, if the relative riskiness between the risky assets is acceptable, the investor will equally invest in them regardless of their correlation. The worst-case scenario of the expected return is its lower bound if the investor takes a long position on the risky asset, while its upper bound if the risky asset is shorted. The worst-case scenario of correlation is its lower bound if any risky asset is shorted; otherwise, the investor will take the upper bound as the conservative belief of the correlation.

In conclusion, this new point of view highlights the effect of the ambiguity about the expected return and correlation between the risky assets, and provides a novel explanation for optimal asset allocation. The resulting investment strategies are consistent with some stylized facts from the cross-hedging point of view, such as “nonparticipation in the risky asset market”, “under-diversification”, and “home bias”.

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A Appendix

In the appendix, we will give the proof of Theorem 3.1. Before giving its proof, we first give two useful lemmas.

Lemma A.1. *Suppose $\varphi \in C^{1,2}\left((0,T)\times\mathbb{R}^+\right)$ with the polynomial growth conditions, and $\theta = (\mu_1,\mu_2,\rho) \in \Theta = \Lambda \times [\underline{\rho},\overline{\rho}]$. Assume the following conditions hold.*

(i) Let φ be a solution of the following equation (Hamilton-Jacobi-Bellman-Isaacs equation),

$$\sup_{\pi}\inf_{\theta\in\Theta}\Big\{\varphi_t(t,x)+x\varphi_x(t,x)\big[\pi\mu_1+(1-\pi)\mu_2\big] \\ +\frac{1}{2}x^2\varphi_{xx}(t,x)\big[\pi^2\sigma_1^2+2\pi(1-\pi)\rho\sigma_1\sigma_2+(1-\pi)^2\sigma_2^2\big]\Big\}=0 \tag{18}$$

with boundary condition

$$\varphi(T,x)=u(T,x)\,.$$

(ii) Let $\hat{\pi}(x)\in\mathbb{R}$ satisfy

$$\hat{\pi}(x)=\arg\sup_{\pi}\inf_{\theta\in\Theta}\Big\{\varphi_t(t,x)+x\varphi_x(t,x)\big[\pi\mu_1+(1-\pi)\mu_2\big] \\ +\frac{1}{2}x^2\varphi_{xx}(t,x)\big[\pi^2\sigma_1^2+2\pi(1-\pi)\rho\sigma_1\sigma_2+(1-\pi)^2\sigma_2^2\big]\Big\}$$

and $(\hat{\mu}_1(x),\hat{\mu}_2(x),\hat{\rho}(x))\in\Theta$ satisfy

$$(\hat{\mu}_1(x),\hat{\mu}_2(x),\hat{\rho}(x))=\arg\inf_{\theta\in\Theta}\Big\{\varphi_t(t,x)+x\varphi_x(t,x)\big[\hat{\pi}\mu_1+(1-\hat{\pi})\mu_2\big] \\ +\frac{1}{2}x^2\varphi_{xx}(t,x)\big[(\hat{\pi})^2\sigma_1^2+2\hat{\pi}(1-\hat{\pi})\rho\sigma_1\sigma_2+(1-\hat{\pi})^2\sigma_2^2\big]\Big\}.$$

(iii) If X^* is the unique solution of the following stochastic differential equation

$$\mathrm{d}X_t^*=X_t^*\big[\hat{\pi}_t(X_t^*)\hat{\mu}_1(X_t^*)+(1-\hat{\pi}(X_t^*))\,\hat{\mu}_2(X_t^*)\big]\mathrm{d}t+X_t^*\big[\hat{\pi}(X_t^*)\sigma_1+(1-\hat{\pi}(X_t^*))\,\hat{\rho}(X_t^*)\sigma_2\big]\mathrm{d}W_{1,t}^{\theta^*} \\ +(1-\hat{\pi}(X_t^*))\sqrt{1-\rho^{*2}}\sigma_2X_t^*\mathrm{d}W_{2,t}^{\theta^*},$$

and $X_0^*=x_0$, where $\theta^*=(\hat{\mu}_1(X^*),\hat{\mu}_2(X^*),\hat{\rho}(X^*))$.

We define $\pi_t^*=\hat{\pi}\,(X_t^*)$, $\mu_{1,t}^*=\hat{\mu}_1(X_t^*),\mu_{2,t}^*=\hat{\mu}_2(X_t^*),\rho_t^*=\hat{\rho}(X_t^*)$, for $t\in[0,T]$. If $\pi^*\in\mathcal{A}(x_0)$, and $(\mu_1^*,\mu_2^*,\rho^*)\in\Gamma^{\Theta}$, then π^* is the optimal investment strategy, and

$$V\,(x_0)=\varphi\,(0,x_0)=\sup_{\pi\in\mathcal{A}(x_0)}\inf_{\mathbb{P}^\theta\in\mathcal{P}^\Theta}\mathbb{E}^{\mathbb{P}^\theta}\left[u\,(T,X_T^\pi)\right]\,.$$

Proof. Since Θ is compact, we know that for any $\pi\in\mathbb{R}$, there exists $\tilde{\theta}=(\tilde{\mu_1},\tilde{\mu}_2,\tilde{\rho})$ such that

$$\inf_{\theta\in\Theta}\Big\{x\varphi_x(t,x)\big[\pi\mu_1+(1-\pi)\mu_2\big]+\frac{1}{2}x^2\varphi_{xx}(t,x)\big[\pi^2\sigma_1^2+2\pi(1-\pi)\rho\sigma_1\sigma_2+(1-\pi)^2\sigma_2^2\big]\Big\} \\ =x\varphi_x(t,x)\big[\pi\tilde{\mu}_1+(1-\pi)\tilde{\mu}_2\big]+\frac{1}{2}x^2\varphi_{xx}(t,x)\big[\pi^2\sigma_1^2+2\pi(1-\pi)\tilde{\rho}\sigma_1\sigma_2+(1-\pi)^2\sigma_2^2\big].$$

For any admissible strategy π, we let $\tilde{X}$ be the wealth process under $P^{(\tilde{\mu_1},\tilde{\mu}_2,\tilde{\rho})}$ as follows:

$$\mathrm{d}\tilde{X}_t=\tilde{X}_t\big[\pi_t\tilde{\mu}_1+(1-\pi_t)\,\tilde{\mu}_2\big]\mathrm{d}t+\tilde{X}_t\big[\pi_t\sigma_1+(1-\pi_t)\,\tilde{\rho}\sigma_2\big]\mathrm{d}W_{1,t}^{\tilde{\mu_1},\tilde{\mu}_2,\tilde{\rho}} \\ +(1-\pi_t)\sqrt{1-\tilde{\rho}^2}\sigma_2\tilde{X}_t\mathrm{d}W_{2,t}^{\tilde{\mu_1},\tilde{\mu}_2,\tilde{\rho}},$$

and $\tilde{X}_0=x_0$.

By Itô's lemma, we have

$$
\begin{aligned}
d\varphi(t,\tilde{X}_t) = \Big(&\varphi_t(t,\tilde{X}_t) + \tilde{X}_t\varphi_x(t,\tilde{X}_t)\big[\pi_t\tilde{\mu}_1 + (1-\pi_t)\tilde{\mu}_2\big] \\
&+ \frac{1}{2}\tilde{X}_t^2\varphi_{xx}(t,\tilde{X}_t)\big[\pi^2\sigma_1^2 + 2\pi_t(1-\pi_t)\tilde{\rho}\sigma_1\sigma_2 + (1-\pi_t)^2\sigma_2^2\big]\Big)dt \\
&+ \varphi_x(t,\tilde{X}_t)\tilde{X}_t\big[\pi_t\sigma_1 + (1-\pi_t)\,\tilde{\rho}\sigma_2\big]\mathrm{d}W_{1,t}^{\tilde{\mu_1},\tilde{\mu}_2,\tilde{\rho}} + \varphi_x(t,\tilde{X}_t)\,(1-\pi_t)\,\sqrt{1-\tilde{\rho}^2}\sigma_2\tilde{X}_t\mathrm{d}W_{2,t}^{\tilde{\mu_1},\tilde{\mu}_2,\tilde{\rho}}.
\end{aligned}
$$

From (18) and (19) we have

$$
d\varphi(t,\tilde{X}_t) \le \varphi_x(t,\tilde{X}_t)\tilde{X}_t\big[\pi_t\sigma_1 + (1-\pi_t)\,\tilde{\rho}\sigma_2\big]\mathrm{d}W_{1,t}^{\tilde{\mu_1},\tilde{\mu}_2,\tilde{\rho}} + \varphi_x(t,\tilde{X}_t)\,(1-\pi_t)\,\sqrt{1-\tilde{\rho}^2}\sigma_2\tilde{X}_t\mathrm{d}W_{2,t}^{\tilde{\mu_1},\tilde{\mu}_2,\tilde{\rho}}.
$$

Therefore, $\mathbb{E}^{P^{(\tilde{\mu_1},\tilde{\mu}_2,\tilde{\rho})}}[u(T,\tilde{X}_T)] = \mathbb{E}^{P^{(\tilde{\mu_1},\tilde{\mu}_2,\tilde{\rho})}}[\varphi(T,\tilde{X}_T)] \le \varphi(0,x_0)$, and

$$
V(x_0) = \sup_{\pi\in\mathcal{A}(x_0)}\inf_{\mathbb{P}^\theta\in\mathcal{P}^\Theta}\mathbb{E}^{\mathbb{P}^\theta}\left[u\left(T,X_T^\pi\right)\right] \le \varphi\left(0,x_0\right).
$$

Let us denote $g(t,\mu_1,\mu_2,\rho) = \varphi_t(t,X_t^{\pi^*}) + X_t^{\pi^*}\varphi_x(t,X_t^{\pi^*})\big[\pi_t^*\mu_1 + (1-\pi_t^*)\mu_2\big] + \frac{1}{2}(X^{\pi^*})_t^2\varphi_{xx}(t,X_t^{\pi^*})\big[(\pi_t^*)^2\sigma_1^2 + 2\pi_t^*(1-\pi_t^*)\rho\sigma_1\sigma_2 + (1-\pi_t^*)^2\sigma_2^2\big]$. Then

$$
\begin{aligned}
V(x_0) &= \sup_{\pi\in\mathcal{A}(x_0)}\inf_{\mathbb{P}^\theta\in\mathcal{P}^\Theta}\mathbb{E}^{\mathbb{P}^\theta}\left[u\left(T,X_T^\pi\right)\right] \\
&\ge \inf_{\mathbb{P}^\theta\in\mathcal{P}^\Theta}\mathbb{E}^{\mathbb{P}^\theta}\left[u\left(T,X_T^{\pi^*}\right)\right] \\
&= \inf_{(\mu_1,\mu_2,\rho)\in\Gamma^\Theta}\mathbb{E}^{\mathbb{P}^{(\mu_1,\mu_2,\rho)}}\left[\int_0^T g(t,\mu_{1,t},\mu_{2,t},\rho_t)dt\right] + \varphi\left(0,x_0\right) \\
&\ge \inf_{(\mu_1,\mu_2,\rho)\in\Gamma^\Theta}\mathbb{E}^{\mathbb{P}^{(\mu_1,\mu_2,\rho)}}\left[\int_0^T \inf_{(\mu_1,\mu_2,\rho)\in\Gamma^\Theta} g(t,\mu_{1,t},\mu_{2,t},\rho_t)dt\right] + \varphi\left(0,x_0\right).
\end{aligned}
$$

Thanks to the assumptions on π^*, μ_1^*, μ_2^*, ρ^*, we have

$$
\begin{aligned}
V(x_0) &\ge \inf_{(\mu_1,\mu_2,\rho)\in\Gamma^\Theta}\mathbb{E}^{\mathbb{P}^{(\mu_1,\mu_2,\rho)}}\left[\int_0^T g(t,\mu_{1,t}^*,\mu_{2,t}^*,\rho_t^*)dt\right] + \varphi\left(0,x_0\right) \\
&= \mathbb{E}^{\mathbb{P}^{(\mu_1^*,\mu_2^*,\rho^*)}}\left[\int_0^T g(t,\mu_{1,t}^*,\mu_{2,t}^*,\rho_t^*)dt\right] + \varphi\left(0,x_0\right) \\
&= \varphi\left(0,x_0\right).
\end{aligned}
$$

From the above we know that π^* is the optimal investment strategy and

$$
V(x_0) = \varphi\left(0,x_0\right) = \sup_{\pi\in\mathcal{A}(x_0)}\inf_{\mathbb{P}^\theta\in\mathcal{P}^\Theta}\mathbb{E}^{\mathbb{P}^\theta}\left[u\left(T,X_T^\pi\right)\right].
$$

□

Lemma A.2. *If $a = \frac{1}{2}x^2\varphi_{xx}(t,x) < 0$, $b = \varphi_x(t,x)x > 0$, and*

$$
f(\pi) = \inf_{\theta\in\Theta}\Big\{b\big[\pi\mu_1 + (1-\pi)\mu_2\big] + a\big[\pi^2\sigma_1^2 + 2\pi(1-\pi)\rho\sigma_1\sigma_2 + (1-\pi)^2\sigma_2^2\big]\Big\},
$$

The optimization problem

$$
\sup_{\pi\in\mathbb{R}} f(\pi)
$$

has the following solution:

(1) If $\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\underline{\mu}_1 - \underline{\mu}_2) = 0$*, then*

$$\sup_\pi f(\pi) = f(\hat{\pi}),$$

where $\hat{\pi} = \frac{1}{2}$.

(2) If $\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\overline{\mu}_1 - \underline{\mu}_2) > 0$.

(i) If $\underline{\rho} < \overline{\rho} < \frac{\sigma_2}{\sigma_1} - \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$*, then*

$$\sup_\pi f(\pi) = f(\hat{\pi}),$$

where $\hat{\pi} = \frac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (0, \frac{1}{2})$.

(ii) If $\underline{\rho} \leq \frac{\sigma_2}{\sigma_1} - \frac{b(\overline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$ *and* $\frac{\sigma_2}{\sigma_1} - \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} \leq \overline{\rho}$*, then*

$$\sup_\pi f(\pi) = f(\hat{\pi}) = b\underline{\mu}_2 + a\sigma_2^2,$$

where $\hat{\pi} = 0$.

(iii) If $\frac{\sigma_2}{\sigma_1} - \frac{b(\overline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} < \underline{\rho} < \overline{\rho}$*, then*

$$\sup_\pi f(\pi) = f(\hat{\pi}),$$

where $\hat{\pi} = \frac{\sigma_2^2 - \underline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \frac{b(\overline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (-\infty, 0)$.

(3) If $\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\underline{\mu}_1 - \overline{\mu}_2) \leq 0$.

(i) If $\underline{\rho} < \overline{\rho} < \frac{\sigma_1}{\sigma_2} + \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$*, then*

$$\sup_\pi f(\pi) = f(\hat{\pi}),$$

where $\hat{\pi} = \frac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (\frac{1}{2}, 1)$.

(ii) If $\underline{\rho} \leq \frac{\sigma_1}{\sigma_2} + \frac{b(\underline{\mu}_1 - \overline{\mu}_2)}{2a\sigma_1\sigma_2}$ *and* $\frac{\sigma_1}{\sigma_2} + \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} \leq \overline{\rho}$*, then*

$$\sup_\pi f(\pi) = f(\hat{\pi}) = b\underline{\mu}_1 + a\sigma_1^2,$$

where $\hat{\pi} = 1$.

(iii) If $\frac{\sigma_1}{\sigma_2} + \frac{b(\underline{\mu}_1 - \overline{\mu}_2)}{2a\sigma_1\sigma_2} < \underline{\rho} < \overline{\rho}$*, then*

$$\sup_\pi f(\pi) = f_3(\hat{\pi}),$$

where $\hat{\pi} = \frac{\sigma_2^2 - \underline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \frac{b(\underline{\mu}_1 - \overline{\mu}_2)}{2a(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (1, +\infty)$.

(4) If $\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\underline{\mu}_1 - \underline{\mu}_2) > 0$ *and* $\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\overline{\mu}_1 - \underline{\mu}_2) \le 0$*.*

(i) If $\frac{\sigma_2}{\sigma_1} - \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} \le \overline{\rho}$*, then*

$$\sup_{\pi} f(\pi) = f(\hat{\pi}) = b\underline{\mu}_2 + a\sigma_2^2,$$

where $\hat{\pi} = 0$*.*

(ii) If $\overline{\rho} < \frac{\sigma_2}{\sigma_1} - \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$*, then*

$$\sup_{\pi} f(\pi) = f(\hat{\pi}),$$

where $\hat{\pi} = \frac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (0, \frac{1}{2})$*.*

(5) If $\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\underline{\mu}_1 - \underline{\mu}_2) < 0$ *and* $\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\underline{\mu}_1 - \overline{\mu}_2) > 0$*.*

(i) If $\frac{\sigma_1}{\sigma_2} + \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} \le \overline{\rho}$*, then*

$$\sup_{\pi} f(\pi) = f(\hat{\pi}) = b\underline{\mu}_1 + a\sigma_1^2,$$

where $\hat{\pi} = 1$*.*

(ii) If $\overline{\rho} < \frac{\sigma_1}{\sigma_2} + \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$*, then*

$$\sup_{\pi} f(\pi) = f(\hat{\pi}),$$

where $\hat{\pi} = \frac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (\frac{1}{2}, 1)$*.*

Proof. Recalling the function f, we define the following functions:

$$\begin{aligned}
f_1(\pi) &= a\pi^2(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2) + b\pi(\underline{\mu}_1 - \underline{\mu}_2) + 2a\pi(\overline{\rho}\sigma_1\sigma_2 - \sigma_2^2) + b\underline{\mu}_2 + a\sigma_2^2\,, \quad 0 \le \pi \le 1\,,\\
f_2(\pi) &= a\pi^2(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2) + b\pi(\overline{\mu}_1 - \underline{\mu}_2) + 2a\pi(\underline{\rho}\sigma_1\sigma_2 - \sigma_2^2) + b\underline{\mu}_2 + a\sigma_2^2\,, \quad \pi \le 0\,,\\
f_3(\pi) &= a\pi^2(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2) + b\pi(\underline{\mu}_1 - \overline{\mu}_2) + 2a\pi(\underline{\rho}\sigma_1\sigma_2 - \sigma_2^2) + b\overline{\mu}_2 + a\sigma_2^2\,, \quad \pi \ge 1\,.
\end{aligned}$$

Then, the optimization problem can be decomposed as the following form

$$\sup_{\pi \in \mathbb{R}} f(\pi) = \sup_{0 \le \pi \le 1} f_1(\pi) \vee \sup_{\pi \le 0} f_2(\pi) \vee \sup_{\pi \ge 1} f_3(\pi)\,.$$

We first consider $\sup_{0 \le \pi \le 1} f_1(\pi)$, and define

$$\begin{aligned}
\pi_1 &= \frac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)}\\
&= \frac{1}{2} - \frac{\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\underline{\mu}_1 - \underline{\mu}_2)}{2(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)}.
\end{aligned}$$

(A1) If $\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\underline{\mu}_1 - \underline{\mu}_2) > 0$, then π_1 is decreasing in $\overline{\rho}$, and $\pi_1 < \frac{1}{2}$.

(i) If $\dfrac{\sigma_2}{\sigma_1} - \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} \le \overline{\rho}$, then $\pi_1 \le 0$. Therefore,

$$\sup_{0\le\pi\le 1} f_1(\pi) = f_1(\hat{\pi}) = f_1(0) = b\underline{\mu}_2 + a\sigma_2^2 > f_1(1) = b\underline{\mu}_1 + a\sigma_1^2,$$

where $\hat{\pi} = 0$.

(ii) If $\overline{\rho} < \dfrac{\sigma_2}{\sigma_1} - \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$, then $\pi_1 \in (0, \frac{1}{2})$. Therefore,

$$\sup_{0\le\pi\le 1} f_1(\pi) = f_1(\hat{\pi}) > f_1(0) = b\underline{\mu}_2 + a\sigma_2^2 > f_1(1) = b\underline{\mu}_1 + a\sigma_1^2,$$

where $\hat{\pi} = \dfrac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (0, \frac{1}{2})$.

(A2) If $\sigma_1^2 - \sigma_2^2 + \dfrac{b}{a}(\underline{\mu}_1 - \underline{\mu}_2) < 0$, then π_1 is increasing in $\overline{\rho}$, and $\pi_1 > \dfrac{1}{2}$.

(i) If $\dfrac{\sigma_1}{\sigma_2} + \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} \le \overline{\rho}$, then $\pi_1 \ge 1$. Therefore,

$$\sup_{0\le\pi\le 1} f_1(\pi) = f_1(\hat{\pi}) = f_1(1) = b\underline{\mu}_1 + a\sigma_1^2 > f_1(0) = b\underline{\mu}_2 + a\sigma_2^2,$$

where $\hat{\pi} = 1$.

(ii) If $\overline{\rho} < \dfrac{\sigma_1}{\sigma_2} + \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$, then $\pi_1 \in (\dfrac{1}{2}, 1)$. Therefore,

$$\sup_{0\le\pi\le 1} f_1(\pi) = f_1(\hat{\pi}) > f_1(1) = b\underline{\mu}_1 + a\sigma_1^2 > f_1(0) = b\underline{\mu}_2 + a\sigma_2^2,$$

where $\hat{\pi} = \dfrac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (\dfrac{1}{2}, 1)$.

(A3) If $\sigma_1^2 - \sigma_2^2 + \dfrac{b}{a}(\underline{\mu}_1 - \underline{\mu}_2) = 0$, then $\pi_1 = \dfrac{1}{2}$.

$$\sup_{0\le\pi\le 1} f_1(\pi) = f_1(\hat{\pi}) > f_1(1) = b\underline{\mu}_1 + a\sigma_1^2 = f_1(0) = b\underline{\mu}_2 + a\sigma_2^2,$$

where $\hat{\pi} = \dfrac{1}{2}$.

We now consider $\sup_{\pi\le 0} f_2(\pi)$, and define

$$\begin{aligned}\pi_2 &= \frac{\sigma_2^2 - \underline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \frac{b(\overline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \\ &= \frac{1}{2} - \frac{\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\overline{\mu}_1 - \underline{\mu}_2)}{2(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2)}.\end{aligned}$$

(B1) If $\sigma_1^2 - \sigma_2^2 + \dfrac{b}{a}(\overline{\mu}_1 - \underline{\mu}_2) > 0$, then π_2 is decreasing in $\underline{\rho}$, and $\pi_2 < \frac{1}{2}$.

(i) If $\dfrac{\sigma_2}{\sigma_1} - \dfrac{b(\overline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} < \underline{\rho}$, then $\pi_2 < 0$. Therefore,

$$\sup_{\pi\le 0} f_2(\pi) = f_2(\hat{\pi}) > f_1(0) = b\underline{\mu}_2 + a\sigma_2^2,$$

where $\hat{\pi} = \dfrac{\sigma_2^2 - \underline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \dfrac{b(\overline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} < 0$.

(ii) If $\underline{\rho} \le \dfrac{\sigma_2}{\sigma_1} - \dfrac{b(\overline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$, then $\pi_2 \in [0, \frac{1}{2})$. Therefore,

$$\sup_{\pi \le 0} f_2(\pi) = f_2(\hat{\pi}) = f_1(0) = b\underline{\mu}_2 + a\sigma_2^2 > f_2(1) = b\overline{\mu}_1 + a\sigma_1^2,$$

where $\hat{\pi} = 0$.

(B2) If $\sigma_1^2 - \sigma_2^2 + \dfrac{b}{a}(\underline{\mu}_1 - \underline{\mu}_2) \le 0$, then π_2 is decreasing in $\underline{\rho}$, and $\pi_2 \ge \frac{1}{2}$. Therefore,

$$\sup_{\pi \le 0} f_2(\pi) = f_2(\hat{\pi}) = f_2(0) = b\underline{\mu}_2 + a\sigma_2^2 \le f_2(1) = b\overline{\mu}_1 + a\sigma_1^2,$$

where $\hat{\pi} = 0$.

We consider $\sup_{\pi \ge 1} f_3(\pi)$, and define

$$\begin{aligned}
\pi_3 &= \frac{\sigma_2^2 - \underline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \frac{b(\underline{\mu}_1 - \overline{\mu}_2)}{2a(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \\
&= \frac{1}{2} - \frac{\sigma_1^2 - \sigma_2^2 + \dfrac{b}{a}(\underline{\mu}_1 - \overline{\mu}_2)}{2(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2)}.
\end{aligned}$$

(C1) If $\sigma_1^2 - \sigma_2^2 + \dfrac{b}{a}(\underline{\mu}_1 - \overline{\mu}_2) > 0$, then $\pi_3 < \dfrac{1}{2}$. Therefore,

$$\sup_{\pi \ge 1} f_3(\pi) = f_3(\hat{\pi}) = f_3(1) = b\underline{\mu}_1 + a\sigma_1^2,$$

where $\hat{\pi} = 1$.

(C2) If $\sigma_1^2 - \sigma_2^2 + \dfrac{b}{a}(\underline{\mu}_1 - \overline{\mu}_2) \le 0$, then π_3 is increasing in $\underline{\rho}$, and $\pi_3 \ge \dfrac{1}{2}$.

(i) If $\underline{\rho} \le \dfrac{\sigma_1}{\sigma_2} + \dfrac{b(\underline{\mu}_1 - \overline{\mu}_2)}{2a\sigma_1\sigma_2}$, then $\pi_3 \in [\frac{1}{2}, 1]$. Therefore,

$$\sup_{\pi \ge 1} f_3(\pi) = f_3(\hat{\pi}) = f_3(1) = b\underline{\mu}_1 + a\sigma_1^2 \ge f_3(0) = b\overline{\mu}_2 + a\sigma_2^2,$$

where $\hat{\pi} = 1$.

(ii) If $\dfrac{\sigma_1}{\sigma_2} + \dfrac{b(\underline{\mu}_1 - \overline{\mu}_2)}{2a\sigma_1\sigma_2} < \underline{\rho}$, then $\pi_3 > 1$. Therefore,

$$\sup_{\pi \ge 1} f_3(\pi) = f_3(\hat{\pi}) > f_3(1) = b\underline{\mu}_1 + a\sigma_1^2,$$

where $\hat{\pi} = \dfrac{\sigma_2^2 - \underline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \dfrac{b(\underline{\mu}_1 - \overline{\mu}_2)}{2a(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} > 1$.

If $\sigma_1^2 - \sigma_2^2 + \dfrac{b}{a}(\overline{\mu}_1 - \underline{\mu}_2) > 0$, then from (A1), (B1) and (C1) we have the following.

(i) If $\underline{\rho} < \overline{\rho} < \dfrac{\sigma_2}{\sigma_1} - \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$, then

$$\sup_{\pi} f(\pi) = f_1(\hat{\pi}) = f(\hat{\pi}),$$

where $\hat{\pi} = \dfrac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (0, \frac{1}{2})$.

(ii) If $\underline{\rho} \leq \frac{\sigma_2}{\sigma_1} - \frac{b(\overline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$ and $\frac{\sigma_2}{\sigma_1} - \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} \leq \overline{\rho}$, then

$$\sup_{\pi} f(\pi) = f_1(\hat{\pi}) = f_1(0) = f_2(0) = b\underline{\mu}_2 + a\sigma_2^2,$$

where $\hat{\pi} = 0$.

(iii) If $\frac{\sigma_2}{\sigma_1} - \frac{b(\overline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} < \underline{\rho} < \overline{\rho}$, then

$$\sup_{\pi} f(\pi) = f_2(\hat{\pi}),$$

where $\hat{\pi} = \frac{\sigma_2^2 - \underline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \frac{b(\overline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (-\infty, 0)$.

If $\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\underline{\mu}_1 - \overline{\mu}_2) \leq 0$, then from (A2), (B2) and (C2) we have the following.

(i) If $\underline{\rho} < \overline{\rho} < \frac{\sigma_1}{\sigma_2} + \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$, then

$$\sup_{\pi} f(\pi) = f_1(\hat{\pi}) = f(\hat{\pi}),$$

where $\hat{\pi} = \frac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (\frac{1}{2}, 1)$.

(ii) If $\underline{\rho} \leq \frac{\sigma_1}{\sigma_2} + \frac{b(\underline{\mu}_1 - \overline{\mu}_2)}{2a\sigma_1\sigma_2}$ and $\frac{\sigma_1}{\sigma_2} + \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} \leq \overline{\rho}$, then

$$\sup_{\pi} f(\pi) = f_1(\hat{\pi}) = f_1(1) = f_3(1) = b\underline{\mu}_1 + a\sigma_1^2,$$

where $\hat{\pi} = 1$.

(iii) If $\frac{\sigma_1}{\sigma_2} + \frac{b(\underline{\mu}_1 - \overline{\mu}_2)}{2a\sigma_1\sigma_2} < \underline{\rho} < \overline{\rho}$, then

$$\sup_{\pi} f(\pi) = f_3(\hat{\pi}),$$

where $\hat{\pi} = \frac{\sigma_2^2 - \underline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \frac{b(\underline{\mu}_1 - \overline{\mu}_2)}{2a(\sigma_1^2 - 2\underline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (1, +\infty)$.

If $\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\underline{\mu}_1 - \underline{\mu}_2) = 0$, then from (A3), (B2) and (C1) it follows that

$$\sup_{\pi} f(\pi) = f_1(\hat{\pi}),$$

where $\hat{\pi} = \frac{1}{2}$.

If $\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\underline{\mu}_1 - \underline{\mu}_2) > 0$ and $\sigma_1^2 - \sigma_2^2 + \frac{b}{a}(\overline{\mu}_1 - \underline{\mu}_2) \leq 0$, then from (A1), (B2) and (C1) we have the following.

(i) If $\frac{\sigma_2}{\sigma_1} - \frac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} \leq \overline{\rho}$, then

$$\sup_{\pi} f(\pi) = f_1(\hat{\pi}) = f_1(0) = b\underline{\mu}_2 + a\sigma_2^2,$$

where $\hat{\pi} = 0$.

(ii) If $\overline{\rho} < \dfrac{\sigma_2}{\sigma_1} - \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$, then

$$\sup_{\pi} f(\pi) = f_1(\hat{\pi}),$$

where $\hat{\pi} = \dfrac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (0, \frac{1}{2})$.

If $\sigma_1^2 - \sigma_2^2 + \dfrac{b}{a}(\underline{\mu}_1 - \underline{\mu}_2) < 0$ and $\sigma_1^2 - \sigma_2^2 + \dfrac{b}{a}(\underline{\mu}_1 - \overline{\mu}_2) > 0$, then from (A2), (B2) and (C1) we have the following.

(i) If $\dfrac{\sigma_1}{\sigma_2} + \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2} \leq \overline{\rho}$, then

$$\sup_{\pi} f(\pi) = f_1(\hat{\pi}) = f_1(1) = b\underline{\mu}_1 + a\sigma_1^2,$$

where $\hat{\pi} = 1$.

(ii) If $\overline{\rho} < \dfrac{\sigma_1}{\sigma_2} + \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a\sigma_1\sigma_2}$, then

$$\sup_{\pi} f(\pi) = f_1(\hat{\pi}),$$

where $\hat{\pi} = \dfrac{\sigma_2^2 - \overline{\rho}\sigma_1\sigma_2}{\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2} - \dfrac{b(\underline{\mu}_1 - \underline{\mu}_2)}{2a(\sigma_1^2 - 2\overline{\rho}\sigma_1\sigma_2 + \sigma_2^2)} \in (\frac{1}{2}, 1)$.

□

Proof of Theorem 3.1.

Proof. Let us suppose

$$\varphi(t, x) = g(t)\frac{x^{(1-\kappa)}}{1-\kappa},$$

where $g(t)$ is a deterministic function of t. $g(t)$ is solved by using Lemmas A.1 and A.2, and we omit it here.

□