Online Appendix for “Long-run Consumption Risk and Asset Allocation under Recursive Utility and Rational Inattention”

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1. Deriving the Stochastic Properties of Consumption Dynamics

Taking unconditional variance on both sides of the expression for individual consumption dynamics,

$$\Delta c_{t+1}^* = \theta \left\{ \frac{\alpha^* u_{t+1}}{1 - ((1 - \theta) / \phi) \cdot L} + \left[ \xi_{t+1} - \frac{(\theta / \phi) \xi_t}{1 - ((1 - \theta) / \phi) \cdot L} \right] \right\},$$  \hspace{1cm} (1)

yields

$$\text{var} \left[ \Delta c_t^* \right] = \theta^2 \left\{ \frac{\alpha^{*2} \omega^2}{1 - (1 - \theta)^2 / \phi^2} + \frac{\theta^2 / \beta^2}{1 - (1 - \theta)^2 / \beta^2} \right\} \omega^2_{\xi}$$

$$= \theta^2 \left\{ \frac{1}{1 - (1 - \theta)^2 / \phi^2} + \frac{1}{(1 - \theta) / \phi - \frac{1}{1 - (1 - \theta)^2 / \phi^2}} \right\} \alpha^{*2} \omega^2$$

$$= \frac{\theta^2 \phi^2}{\phi^2 + \theta - 1} \alpha^{*2} \omega^2.$$

Note that in the absence of the endogenous noise shocks (i.e., $\xi_t = 0$), we have

$$\text{var} \left[ \Delta c_t^* \right] = \theta^2 \frac{\alpha^{*2} \omega^2}{1 - (1 - \theta)^2 / \phi^2} = \frac{\theta^2}{1 - (1 - \theta) / \phi^2} \alpha^{*2} \omega^2.$$

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Using (1), we can compute the first-order autocovariance of consumption growth as follows:

$$\text{cov}(\Delta c_t^*, \Delta c_{t+1}^*) = \text{cov} \left( \theta \left\{ \frac{\theta^* u_t}{1 - ((1 - \theta) / \phi) \cdot L} + \frac{\xi_t - ((\theta / \phi) \xi_{t-1})}{1 - ((1 - \theta) / \phi) \cdot L} \right\}, \theta \left\{ \frac{\alpha^* u_{t+1}}{1 - ((1 - \theta) / \phi) \cdot L} + \frac{\xi_{t+1} - ((\theta / \phi) \xi_t)}{1 - ((1 - \theta) / \phi) \cdot L} \right\} \right)$$

$$= \frac{1 - \theta}{\phi} \text{cov} \left( \frac{\theta \alpha^* u_t}{1 - ((1 - \theta) / \phi) \cdot L}, \frac{\theta \alpha^* u_t}{1 - ((1 - \theta) / \phi) \cdot L} \right)$$

$$+ \text{cov} \left( \theta \left( \xi_t - \frac{(\theta / \phi) \xi_{t-1}}{1 - ((1 - \theta) / \phi) \cdot L} \right), \frac{\theta (\theta / \phi) \xi_t}{1 - ((1 - \theta) / \phi) \cdot L} \right)$$

$$= \frac{1 - \theta}{\phi} \text{cov} \left( \frac{\theta \alpha^* u_t}{1 - ((1 - \theta) / \phi) \cdot L}, \frac{\theta \alpha^* u_t}{1 - ((1 - \theta) / \phi) \cdot L} \right)$$

$$+ \frac{1 - \theta}{\phi} \text{var} \left( \frac{\theta \alpha^* u_t}{1 - ((1 - \theta) / \phi) \cdot L} \right)$$

$$= \frac{1 - \theta}{\phi} \left( \frac{\theta^2 (\theta^* \alpha^2) \omega^2}{1 - (1 - \theta)^2 / \phi^2} \right) + \frac{1 - \theta}{\phi} \left( \frac{\theta^2 (\theta^* \alpha^2) \omega^2}{1 - (1 - \theta)^2 / \phi^2} \right)$$

$$= \frac{1 - \theta}{\phi} \left( \frac{\theta^3 + \theta^2}{1 - (1 - \theta)^2 / \phi^2} \right)$$

$$= 0.$$

We thus have

$$\text{corr}(\Delta c_t^*, \Delta c_{t+1}^*) = \frac{\text{cov}(\Delta c_t^*, \Delta c_{t+1}^*)}{\sqrt{\text{var}(\Delta c_t^*) \sqrt{\text{var}(\Delta c_{t+1}^*)}}} = 0.$$
Finally, using (1), it is straightforward to show that
\[
\text{corr} (\Delta c^*, t+1, u_{t+1}) = \frac{\text{cov} (\Delta c^*, u_{t+1})}{\text{sd}(\Delta c^*) \text{sd}(u_{t+1})} = \frac{\theta \alpha^* \omega^2}{\sqrt{\frac{\theta^2}{\phi^2} + \alpha^* \omega^2}} = \sqrt{\theta \left(1 - \frac{1}{\phi^2}\right)} ,
\]
where we use the result that
\[
\text{cov} (\Delta c^*, u_{t+1}) = \text{cov} (\theta \alpha^* u_{t+1}, u_{t+1}) = \theta \alpha^* \omega^2 .
\]
Note that in the absence of the endogenous noise shocks, we have
\[
\text{corr} (\Delta c^*, u_{t+1}) = \sqrt{\left(1 - \frac{1}{\phi^2}\right)} ,
\]

2. Deriving the Long-run Risk in the Presence of the Correlated Noise

Substituting (1) into
\[
\pi = \text{cov}_t \left\{ \frac{\rho}{\sigma} \left( \sum_{j=0}^{S} \Delta c_{t+1+j} \right) + (1 - \rho) \left( \sum_{j=0}^{S} r_{p,t+1+j} \right), u_{t+1} \right\} ,
\]
we have
\[
\pi = \lim_{S \to \infty} \left\{ \sum_{j=0}^{S} \text{cov}_t \left[ \frac{\rho}{\sigma} \left\{ \frac{u_{t+1+j}}{1 - \frac{1}{(1-\theta)/\phi} L} + \left[ \xi_{t+1+j} - \frac{(\theta/\phi) \xi_{t+1+j}}{1 - \frac{1}{(1-\theta)/\phi} L} \right] \right\} + (1 - \rho) \left( \sum_{j=0}^{S} r_{p,t+1+j} \right), u_{t+1} \right] \right\}
\]
\[
= \frac{\rho \theta}{\sigma} \left\{ \frac{1}{1 - \frac{1}{(1-\theta)/\phi}} \alpha^2 \omega^2 + \left[1 - \frac{\theta}{\phi} \frac{1}{1 - \frac{1}{(1-\theta)/\phi}} \right] \alpha^2 \omega^2 \right\} + (1 - \rho) \alpha^2 \omega^2
\]
\[
= \frac{\rho \theta}{\sigma} \left\{ \frac{1}{1 - \frac{1}{(1-\theta)/\phi}} + \rho \omega^2 \left[1 - \frac{\theta}{\phi} \frac{1}{1 - \frac{1}{(1-\theta)/\phi}} \right] \sqrt{\frac{1}{1 - \frac{1}{(1-\theta)/\phi} \frac{1}{\phi}}} \alpha^2 \omega^2 \right\} + (1 - \rho) \alpha^2 \omega^2
\]
\[
= \left\{ \frac{\rho}{\sigma} \left[1 - \frac{\theta}{\phi} \frac{1}{1 - \frac{1}{(1-\theta)/\phi}} \right] + (1 - \rho) \right\} + \frac{\rho \omega^2 \theta}{\sigma} \left[1 - \frac{\theta}{\phi} \frac{1}{1 - \frac{1}{(1-\theta)/\phi}} \right] \sqrt{\frac{1}{1 - \frac{1}{(1-\theta)/\phi} \frac{1}{\phi}}} \alpha^2 \omega^2 \right\} + (1 - \rho) \alpha^2 \omega^2 ,
\]
which will reduce to the expression for \( \Gamma \) in the text. Note that here we use the fact that \( \omega^2_s = \sqrt{\frac{1}{\left[\exp(2\kappa) - (1/\phi)^2\right] \phi}} \) \( \alpha^2 \omega \).
3. Deriving Optimal Consumption and Portfolio Rules in the Presence of Uninsurable Labor Income

Log-linearizing the flow budget constraint, \( A_{t+1} = R_{t+1}^p \left( A_t + Y_t - C_t \right) \), around the long-run means of the log consumption-income ratio and the log wealth-income ratio, \( c - y = E [c_t - y_t] \) and \( a - y = E [a_t - y_t] \), yields the approximate budget constraint

\[
a_{t+1} - y_{t+1} = \rho_0 + \rho_a (a_t - y_t) + \rho_c (c_t - y_t) - \Delta y_{t+1} + r_{t+1}^p
\]

where \( \rho, \rho_a, \) and \( \rho_c \) are constants:

\[
\rho_a = \frac{\exp (a - y)}{1 + \exp (a - y) - \exp (c - y)} > 0,
\]

\[
\rho_c = \frac{\exp (c - y)}{1 + \exp (a - y) - \exp (c - y)} > 0,
\]

\[
\rho_0 = - (1 - \rho_a + \rho_c) \log (1 - \rho_a + \rho_c) - \rho_a \log (\rho_a) + \rho_c \log (\rho_c) .
\]

Starting from the standard full-information rational expectations model with expected utility \( (\gamma = \sigma \) and \( \theta = 1) \), we obtain the decision rules

\[
c_t = y_t + b_0 + b_1 (a_t - y_t)
\]

\[
\alpha^* = \frac{1}{b_1} \left( \frac{\mu - r_f + \frac{1}{2} \omega^2}{\gamma \omega^2} \right) + \left( 1 - \frac{1}{b_1} \right) \frac{\omega_{ww}}{\omega^2}
\]

where

\[
b_1 = \frac{\rho_a - 1}{\rho_c} \in (0, 1],
\]

\[
b_0 = \frac{1}{1 - \rho_a} \left[ \left( \frac{1}{\gamma} - b_1 \right) E [r_{t+1}^p] + \frac{1}{\gamma} \log (\beta) + \frac{\Xi}{2 \gamma} - \rho_0 - (1 - b_1) g \right].
\]

Here \( b_1 \) is the elasticity of consumption with respect to financial wealth, making \( 1 - b_1 \) the elasticity with respect to labor income, and \( \Xi \) is an irrelevant constant term. If labor income is tradable, \( b_1 = 1 \) and the model reduces to the one studied previously.

To help introduce rational inattention, we define a new state variable

\[
s_t = a_t + \lambda y_t,
\]

where

\[
\lambda = \frac{1 - \rho_a + \rho_c}{\rho_a - 1}.
\]

(As we have noted earlier, multivariate rational inattention models are analytically intractable, so the reduction of the state space to a single variable is critical for our results.) Using this new state
variable, the log-linearized budget constraint (3) can be rewritten as

\[ s_{t+1} = \rho_0 + \rho_a s_t - \rho_c c_t - g + \rho_t \epsilon_{t+1} + \lambda \nu_{t+1} + r^p_{t+1}. \]  

(5)

The consumption function can thus be rewritten as:

\[ c_t = b_0 + b_1 s_t. \]  

(6)

As in the benchmark model, applying the separation principle yields:

\[ c_t = b_0 + b_1 \hat{s}_t \]  

(7)

and we obtain the law of motion for the conditional mean of permanent income

\[ \hat{s}_{t+1} = (1 - \theta) \hat{s}_t + \theta (s_{t+1} + \tilde{g}_{t+1}) + Y, \]  

(8)

where \( Y \) is an irrelevant constant and all other notation is the same as that used in the benchmark model.

Given the assumption that \( 1 - (1 - \theta) \rho_a > 0 \), applying the same long-term Euler equation we used in the benchmark model, we can solve for the optimal share invested in equity in the presence of labor income as:

\[ \alpha^* = \frac{1}{\tilde{\zeta}} \left[ \frac{1}{b_1} \left( \frac{\mu - r^f + 0.5 \omega^2}{\omega^2} \right) + \left( 1 - \frac{1}{b_1} \right) \tilde{\zeta} \omega \omega' \right] \]  

(9)

where \( \tilde{\zeta} = \frac{b_1}{\rho} + 1 - \rho \) and \( \zeta = \frac{\theta}{1 - (1 - \theta) \rho_a} > 1. \)