Online Appendix for "Long-run Consumption Risk and Asset Allocation under Recursive Utility and Rational Inattention"

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1. Deriving the Stochastic Properties of Consumption Dynamics

Taking unconditional variance on both sides of the expression for individual consumption dynamics,

$$\Delta c_{t+1}^{*} = \theta \left\{ \frac{\alpha^{*} u_{t+1}}{1-((1-\theta)/\phi)\cdot L} + \left[\xi_{t+1} - \frac{(\theta/\phi)\,\xi_{t}}{1-((1-\theta)/\phi)\cdot L} \right] \right\}, \tag{1}$$

yields

$$\begin{aligned}
\operatorname{var}\left[\Delta c_{t}^{*}\right] &= \theta^{2} \left\{ \frac{\alpha^{*2}\omega^{2}}{1-(1-\theta)^{2}/\phi^{2}} + \left[1 + \frac{\theta^{2}/\beta^{2}}{1-(1-\theta)^{2}/\beta^{2}} \right] \omega_{\xi}^{2} \right\} \\
&= \theta^{2} \left\{ \frac{1}{1-(1-\theta)^{2}/\phi^{2}} + \left[\frac{1}{(1-(1-\theta)/\phi^{2})\,\theta} - \frac{1}{1-(1-\theta)^{2}/\phi^{2}} \right] \right\} \alpha^{*2}\omega^{2} \\
&= \frac{\theta\phi^{2}}{\phi^{2}+\theta-1}\alpha^{*2}\omega^{2}.
\end{aligned}$$

Note that in the absence of the endogenous noise shocks (i.e., $\xi_t = 0$), we have

$$\operatorname{var}\left[\Delta c_{t}^{*}\right] = \theta^{2}\frac{\alpha^{*2}\omega^{2}}{1-(1-\theta)^{2}/\phi^{2}} = \frac{\theta^{2}}{1-(1-\theta)/\phi^{2}}\alpha^{*2}\omega^{2}.$$

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Using (1), we can compute the first-order autocovariance of consumption growth as follows:

$$
\begin{aligned}
&\operatorname{cov}\left(\Delta c_t^*, \Delta c_{t+1}^*\right) = \operatorname{cov}\left(\begin{array}{c} \theta\left\{\frac{\alpha^* u_t}{1-((1-\theta)/\phi)\cdot L} + \left[\xi_t - \frac{(\theta/\phi)\xi_{t-1}}{1-((1-\theta)/\phi)\cdot L}\right]\right\}, \\ \theta\left\{\frac{\alpha^* u_{t+1}}{1-((1-\theta)/\phi)\cdot L} + \left[\xi_{t+1} - \frac{(\theta/\phi)\xi_t}{1-((1-\theta)/\phi)\cdot L}\right]\right\} \end{array}\right) \\
&= \frac{1-\theta}{\phi}\operatorname{cov}\left(\frac{\theta\alpha^* u_t}{1-((1-\theta)/\phi)\cdot L}, \frac{\theta\alpha^* u_t}{1-((1-\theta)/\phi)\cdot L}\right) \\
&+ \operatorname{cov}\left(\theta\left[\xi_t - \frac{(\theta/\phi)\,\xi_{t-1}}{1-((1-\theta)/\phi)\cdot L}\right], -\frac{\theta\,(\theta/\phi)\,\xi_t}{1-((1-\theta)/\phi)\cdot L}\right) \\
&= \frac{1-\theta}{\phi}\operatorname{var}\left(\frac{\theta\alpha^* u_t}{1-((1-\theta)/\phi)\cdot L}\right) + \operatorname{cov}\left(\theta\left[\xi_t - \frac{(\theta/\phi)\,\xi_{t-1}}{1-((1-\theta)/\phi)\cdot L}\right], -\frac{\theta\,(\theta/\phi)\,\xi_t}{1-((1-\theta)/\phi)\cdot L}\right) \\
&= \frac{1-\theta}{\phi}\frac{(\theta\alpha^*)^2\,\omega^2}{1-(1-\theta)^2/\phi^2} - \frac{\theta^3}{\phi}\frac{\alpha^{*2}\omega^2}{(1/(1-\theta)-1/\phi^2)\,\theta} \\
&+ \frac{\theta^4(1-\theta)}{\phi^3}\frac{1}{1-(1-\theta)^2/\phi^2}\frac{\alpha^{*2}\omega^2}{(1/(1-\theta)-1/\phi^2)\,\theta} \\
&= \left[\frac{1-\theta}{\phi}\frac{\theta^2}{1-(1-\theta)^2/\phi^2} - \frac{\theta^2}{\phi}\frac{1}{1/(1-\theta)-1/\phi^2} + \frac{\theta^3(1-\theta)}{\phi^3}\frac{1}{1-(1-\theta)^2/\phi^2}\frac{1}{1/(1-\theta)-1/\phi^2}\right]\alpha^{*2}\omega^2 \\
&= 0.
\end{aligned}
$$

We thus have

$$
\operatorname{corr}\left(\Delta c_t^*, \Delta c_{t+1}^*\right) = \frac{\operatorname{cov}\left(\Delta c_t^*, \Delta c_{t+1}^*\right)}{\sqrt{\operatorname{var}\left(\Delta c_t^*\right)}\sqrt{\operatorname{var}\left(\Delta c_{t+1}^*\right)}} = 0.
$$

Note that in the absence of the endogenous noise shocks, we have

$$
\begin{aligned}
\operatorname{corr}\left(\Delta c_t^*, \Delta c_{t+1}^*\right) &= \frac{1-\theta}{\phi}\frac{(\theta\alpha^*)^2\,\omega^2}{1-(1-\theta)^2/\phi^2}\left(\frac{\theta\phi^2}{\phi^2+\theta-1}\alpha^{*2}\omega^2\right)^{-1} \\
&= \frac{1-\theta}{\phi}\frac{\theta^2}{1-(1-\theta)^2/\phi^2}\left(\frac{\theta}{1-(1-\theta)^2/\phi^2}\right)^{-1} = \frac{\theta(1-\theta)}{\phi} > 0
\end{aligned}
$$

because

$$
\begin{aligned}
&\operatorname{cov}\left(\Delta c_t^*, \Delta c_{t+1}^*\right) = \operatorname{cov}\left(\frac{\theta\alpha^* u_t}{1-((1-\theta)/\phi)\cdot L}, \frac{\theta\,((1-\theta)/\phi)\,\alpha^* u_t}{1-((1-\theta)/\phi)\cdot L}\right) \\
&= \frac{1-\theta}{\phi}\operatorname{cov}\left(\frac{\theta\alpha^* u_t}{1-((1-\theta)/\phi)\cdot L}, \frac{\theta\alpha^* u_t}{1-((1-\theta)/\phi)\cdot L}\right) \\
&= \frac{1-\theta}{\phi}\frac{(\theta\alpha^*)^2\,\omega^2}{1-(1-\theta)^2/\phi^2}.
\end{aligned}
$$

Finally, using (1), it is straightforward to show that

$$\begin{aligned}
\text{corr}\left(\Delta c^*_{t+1}, u_{t+1}\right) &= \frac{\text{cov}\left(\Delta c^*_{t+1}, u_{t+1}\right)}{\text{sd}\left(\Delta c^*_{t+1}\right)\text{sd}\left(u_{t+1}\right)} \\
&= \frac{\theta\alpha^*\omega^2}{\sqrt{\frac{\theta\phi^2}{\phi^2+\theta-1}\alpha^*\omega^2}} \\
&= \sqrt{\theta\left(1-\left(1-\theta\right)/\phi^2\right)},
\end{aligned}$$

where we use the result that

$$\text{cov}\left(\Delta c^*_{t+1}, u_{t+1}\right) = \text{cov}\left(\theta\alpha^* u_{t+1}, u_{t+1}\right) = \theta\alpha^*\omega^2.$$

Note that in the absence of the endogenous noise shocks, we have

$$\text{corr}\left(\Delta c^*_{t+1}, u_{t+1}\right) = \sqrt{\left(1-\left(1-\theta\right)/\phi^2\right)},$$

2. Deriving the Long-run Risk in the Presence of the Correlated Noise

Substituting (1) into

$$\pi = \text{cov}_t\left[\frac{\rho}{\sigma}\left(\sum_{j=0}^{S}\Delta c_{t+1+j}\right) + \left(1-\rho\right)\left(\sum_{j=0}^{S} r_{p,t+1+j}\right), u_{t+1}\right], \tag{2}$$

we have

$$\begin{aligned}
\pi &= \lim_{S\to\infty}\left\{\sum_{j=0}^{S}\text{cov}_t\left[\begin{array}{c}\frac{\rho}{\sigma}\theta\left\{\frac{\alpha u_{t+1+j}}{1-((1-\theta)/\phi)\cdot L} + \left[\xi_{t+1+j} - \frac{(\theta/\phi)\xi_{t+j}}{1-((1-\theta)/\phi)\cdot L}\right]\right\} \\ +\left(1-\rho\right)\left(\sum_{j=0}^{S} r_{p,t+1+j}\right), u_{t+1}\end{array}\right]\right\} \\
&= \frac{\rho}{\sigma}\theta\left\{\frac{1}{1-\left(1-\theta\right)/\phi}\alpha\omega^2 + \left[1-\frac{\theta}{\phi}\frac{1}{1-\left(1-\theta\right)/\phi}\right]\rho\omega\omega_\xi\right\} + \left(1-\rho\right)\alpha\omega^2 \\
&= \frac{\rho}{\sigma}\theta\left\{\frac{1}{1-\left(1-\theta\right)/\phi} + \rho_{u\xi}\left[1-\frac{\theta}{\phi}\frac{1}{1-\left(1-\theta\right)/\phi}\right]\sqrt{\frac{1}{\left[1/\left(1-\theta\right)-\left(1/\phi\right)^2\right]\theta}}\right\}\alpha\omega^2 + \left(1-\rho\right)\alpha\omega^2 \\
&= \left\{\left[\frac{\rho}{\sigma}\frac{\theta}{1-\left(1-\theta\right)/\phi} + \left(1-\rho\right)\right] + \frac{\rho\rho_{u\xi}}{\sigma}\theta\left[1-\frac{\theta}{\phi}\frac{1}{1-\left(1-\theta\right)/\phi}\right]\sqrt{\frac{1}{\left[1/\left(1-\theta\right)-\left(1/\phi\right)^2\right]\theta}}\right\}\alpha\omega^2,
\end{aligned}$$

which will reduce to the expression for Γ in the text. Note that here we use the fact that $\omega_\xi = \sqrt{\frac{1}{\left[\exp(2\kappa)-(1/\phi)^2\right]\theta}}\alpha\omega$.

3. Deriving Optimal Consumption and Portfolio Rules in the Presence of Uninsurable Labor Income

Log-linearizing the flow budget constraint, $A_{t+1} = R^{p}_{t+1}(A_t + Y_t - C_t)$, around the long-run means of the log consumption-income ratio and the log wealth-income ratio, $c - y = E[c_t - y_t]$ and $a - y = E[a_t - y_t]$, yields the approximate budget constraint

$$a_{t+1} - y_{t+1} = \rho_0 + \rho_a(a_t - y_t) + \rho_c(c_t - y_t) - \Delta y_{t+1} + r^{p}_{t+1} \tag{3}$$

where ρ, ρ_a, and ρ_c are constants:

$$\begin{aligned}
\rho_a &= \frac{\exp(a-y)}{1+\exp(a-y)-\exp(c-y)} > 0, \\
\rho_c &= \frac{\exp(c-y)}{1+\exp(a-y)-\exp(c-y)} > 0, \\
\rho_0 &= -(1-\rho_a+\rho_c)\log(1-\rho_a+\rho_c) - \rho_a\log(\rho_a) + \rho_c\log(\rho_c).
\end{aligned}$$

Starting from the standard full-information rational expectations model with expected utility ($\gamma = \sigma$ and $\theta = 1$), we obtain the decision rules

$$\begin{aligned}
c_t &= y_t + b_0 + b_1(a_t - y_t) \\
\alpha^* &= \frac{1}{b_1}\left(\frac{\mu - r_f + \frac{1}{2}\omega^2}{\gamma\omega^2}\right) + \left(1 - \frac{1}{b_1}\right)\frac{\omega_{uv}}{\omega^2}
\end{aligned}$$

where

$$\begin{aligned}
b_1 &= \frac{\rho_a - 1}{\rho_c} \in (0,1], \\
b_0 &= \frac{1}{1-\rho_a}\left[\left(\frac{1}{\gamma} - b_1\right)E\left[r^{p}_{t+1}\right] + \frac{1}{\gamma}\log(\beta) + \frac{\Xi}{2\gamma} - \rho_0 - (1-b_1)g\right].
\end{aligned}$$

Here b_1 is the elasticity of consumption with respect to financial wealth, making $1 - b_1$ the elasticity with respect to labor income, and Ξ is an irrelevant constant term. If labor income is tradable, $b_1 = 1$ and the model reduces to the one studied previously.

To help introduce rational inattention, we define a new state variable

$$s_t = a_t + \lambda y_t, \tag{4}$$

where

$$\lambda = \frac{1 - \rho_a + \rho_c}{\rho_a - 1}.$$

(As we have noted earlier, multivariate rational inattention models are analytically intractable, so the reduction of the state space to a single variable is critical for our results.) Using this new state variable, the log-linearized budget constraint (3) can be rewritten as

$$s_{t+1} = \rho_0 + \rho_a s_t - \rho_c c_t - g + \rho_\varepsilon \varepsilon_{t+1} + \lambda \nu_{t+1} + r^p_{t+1}. \tag{5}$$

The consumption function can thus be rewritten as:

$$c_t = b_0 + b_1 s_t. \tag{6}$$

As in the benchmark model, applying the separation principle yields:

$$c_t = b_0 + b_1 \widehat{s}_t \tag{7}$$

and we obtain the law of motion for the conditional mean of permanent income

$$\widehat{s}_{t+1} = (1-\theta)\, \widehat{s}_t + \theta\, (s_{t+1} + \xi_{t+1}) + \Upsilon, \tag{8}$$

where Υ is an irrelevant constant and all other notation is the same as that used in the benchmark model.

Given the assumption that $1 - (1-\theta)\, \rho_a > 0$, applying the same long-term Euler equation we used in the benchmark model, we can solve for the optimal share invested in equity in the presence of labor income as:

$$\alpha^* = \frac{1}{\widetilde{\varsigma}} \left[\frac{1}{b_1} \left(\frac{\mu - r^f + 0.5\omega^2}{\omega^2} \right) + \left(1 - \frac{1}{b_1} \right) \frac{\widetilde{\varsigma}\omega_{uv}}{\omega^2} \right] \tag{9}$$

where $\widetilde{\varsigma} = \frac{\rho}{\sigma}\varsigma + 1 - \rho$ and $\varsigma = \frac{\theta}{1-(1-\theta)\rho_a} > 1$.