Online Appendix for "Ignorance, Uncertainty, and Strategic Consumption-Portfolio Decisions" (Not for Publication)

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1 Online Appendix A: Calibrating the Robustness Parameter

The value of the detection error probability, q is determined by the following procedure. Let model P denote the approximating model, ($dS_t = \Lambda dt + \sigma dB_t$), and model Q be the distorted model. Define q_P as

$$q_P = \text{Prob}\left(\ln\left(\frac{L_Q}{L_P}\right) > 0 \middle| P, \mathcal{F}_0^y \right), \tag{1}$$

where $\ln\left(\frac{L_Q}{L_P}\right)$ is the log-likelihood ratio. When model P generates the data, q_P measures the probability that a likelihood ratio test selects model Q. In this case, we call q_P the probability of the model detection error. Similarly, when model Q generates the data, we can define q_Q as

$$q_Q = \text{Prob}\left(\ln\left(\frac{L_P}{L_Q}\right) > 0 \middle| Q, \mathcal{F}_0^y \right) = \text{Prob}\left(\ln\left(\frac{L_Q}{L_P}\right) < 0 \middle| Q, \mathcal{F}_0^y \right). \tag{2}$$

Given initial priors of 0.5 on each model and that the length of the sample is N, the detection error probability, q, can be written as:

$$q(\vartheta; N) = \frac{1}{2}(q_P + q_Q), \tag{3}$$

where ϑ is the robustness parameter used to generate model Q. Given this definition, we can see that $1 - q$ measures the probability that econometricians can distinguish the approximating model from the distorted model. The general idea of the calibration procedure is to find a value of ϑ such that $q(\vartheta; N)$ equals a given value (for example, 10%) after simulating model P and model Q.[1]

We use the method of Maenhout (2006) to compute the detection-error probabilities. Let $\xi_t = \frac{L_Q}{L_P}$, then

$$\ln \xi_t = \ln\left(\frac{L_Q}{L_P}\right) = -\int_0^t \left(\sigma^T v_t\right)^T dB_s - \frac{1}{2}\int_0^t \left\|\sigma^T v_t\right\|^2 ds, \tag{4}$$

[1] The number of periods used in the calculation, N, is set to be 31, the actual length of the data $(1980 - 2010)$.

where

$$\sigma^T v_t = -\vartheta \begin{pmatrix} \frac{\pi}{(\psi\gamma+\vartheta)\sigma_e} \\ \frac{r}{\psi}\sqrt{1-\rho_{ey}^2}\left(\frac{\sigma_y}{r+\rho} + \frac{1}{\sigma_y}\delta p(1-p)f'(p)\right) \end{pmatrix}. \tag{5}$$

The two probabilities in the definition of $q(\vartheta; N)$ can be obtained by finding the characteristic functions of $\ln \xi_N$ in model P and model Q, denoted by $\phi_P(x,t,N)$ and $\phi_Q(x,t,N)$ respectively:

$$\phi_P(x,t,N) = E^P\left[e^{ix\ln\xi_N}\middle| \mathcal{F}_t^y\right] = E^P\left[\xi_N^{ix}\middle| \mathcal{F}_t^y\right],$$
$$\phi_Q(x,t,N) = E^Q\left[e^{ix\ln\xi_N}\middle| \mathcal{F}_t^y\right] = E^P\left[e^{ix\ln\xi_N}\cdot\xi_N\middle| \mathcal{F}_t^y\right] = E^P\left[\xi_N^{ix+1}\middle| \mathcal{F}_t^y\right].$$

From the Feynman-Kac theorem $\phi_P(x,t,N)$ solves the following PDE:

$$\begin{aligned} 0 = {} & \frac{\partial\phi_P}{\partial t} + \left[\lambda_2 - (\lambda_1+\lambda_2)p\right]\frac{\partial\phi_P}{\partial p} + \frac{1}{2}\left\|\sigma^T v_t\right\|^2 \xi_t^2 \frac{\partial^2\phi_P}{\partial\xi_t^2} \\ & - \left(\rho_{ey}\ \sqrt{1-\rho_{ey}^2}\right)\sigma^T v_t \frac{1}{\sigma_y}\delta p(1-p)\xi_t\frac{\partial^2\phi_P}{\partial\xi_t\partial p} + \frac{1}{2\sigma_y^2}\delta^2 p^2(1-p)^2\frac{\partial^2\phi_P}{\partial p^2} \end{aligned}$$

subject to

$$\phi_P(x,N,N) = \xi_N^{ix}.$$

Similarly, the PDE for $\phi_Q(x,t,N)$ is

$$\begin{aligned} 0 = {} & \frac{\partial\phi_Q}{\partial t} + \left[\lambda_2 - (\lambda_1+\lambda_2)p\right]\frac{\partial\phi_Q}{\partial p} + \frac{1}{2}\left\|\sigma^T v_t\right\|^2 \xi_t^2 \frac{\partial^2\phi_Q}{\partial\xi_t^2} \\ & - \left(\rho_{ey}\ \sqrt{1-\rho_{ey}^2}\right)\sigma^T v_t \frac{1}{\sigma_y}\delta p(1-p)\xi_t\frac{\partial^2\phi_Q}{\partial\xi_t\partial p} + \frac{1}{2\sigma_y^2}\delta^2 p^2(1-p)^2\frac{\partial^2\phi_Q}{\partial p^2}, \end{aligned}$$

subject to a different boundary condition: $\phi_Q(x,N,N) = \xi_N^{ix+1}$. Using the Lévy inversion formula, $q_P = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\phi_P(x,0,N)}{ix}\right]dx$ and $q_Q = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\phi_Q(x,0,N)}{ix}\right]dx$, the detection error probability can be written as

$$q(\vartheta; N) = \frac{1}{2} - \frac{1}{2\pi}\int_0^\infty \left(\mathrm{Re}\left[\frac{\phi_Q(x,0,N)}{ix}\right] - \mathrm{Re}\left[\frac{\phi_P(x,0,N)}{ix}\right]\right)dx. \tag{6}$$

The approximation solutions of $\phi_P(x,t,N)$ and $\phi_Q(x,t,N)$ are given by

$$\phi_P(x,t,N) \approx \xi_t^{ix}\exp\left\{ix(ix-1)\left(\frac{1}{2}\left\|\sigma^T v_0\right\|^2(N-t) + \left[A_2(t)p^2 + A_1(t)p + A_0(t)\right]D\delta^2\right)\right\},$$

$$\phi_Q(x,t,N) \approx \xi_t^{ix+1}\exp\left\{ix(ix+1)\left(\frac{1}{2}\left\|\sigma^T v_0\right\|^2(N-t) + \left[A_2(t)p^2 + A_1(t)p + A_0(t)\right]D\delta^2\right)\right\},$$

where

$$A_2(t) = \frac{1}{2(\lambda_1+\lambda_2)}\left[e^{-2(\lambda_1+\lambda_2)(N-t)}-1\right],$$
$$A_1(t) = -\frac{1}{(\lambda_1+\lambda_2)^2}\left[\lambda_2 e^{-2(\lambda_1+\lambda_2)(N-t)}+(\lambda_1-\lambda_2)e^{-(\lambda_1+\lambda_2)(N-t)}-\lambda_1\right],$$
$$A_0(t) = -\frac{\lambda_2}{(\lambda_1+\lambda_2)^2}\left[\frac{\lambda_2}{2(\lambda_1+\lambda_2)}\left(e^{-2(\lambda_1+\lambda_2)(N-t)}-1\right)+\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}\left(e^{-(\lambda_1+\lambda_2)(N-t)}-1\right)+\lambda_1(N-t)\right],$$
$$D = \frac{\vartheta^2 r^2\left(1-\rho_{ey}^2\right)}{\psi^2(r+\rho)^2(r+\lambda_1+\lambda_2)}, \sigma^T v_0 = -\vartheta\begin{pmatrix}\frac{\pi}{(\psi\gamma+\vartheta)\sigma_e} \\ \frac{r}{\psi}\sqrt{1-\rho_{ey}^2}\left(\frac{\sigma_y}{r+\rho}\right)\end{pmatrix}.$$

Then the approximation detection error probability is

$$q(\vartheta;N) \approx \frac{1}{2}-\frac{1}{\pi}\int_0^\infty \frac{r_1(x)\sin\theta_1(x)}{x}dx, \tag{7}$$

where

$$r_1(x) = \exp\left\{-x^2\left[\frac{1}{2}\left\|\sigma^T v_0\right\|^2 N+\left[A_2(0)p_0^2+A_1(0)p_0+A_0(0)\right]D\delta^2\right]\right\},$$
$$\theta_1(x) = x\left(\frac{1}{2}\left\|\sigma^T v_0\right\|^2 N+\left[A_2(0)p_0^2+A_1(0)p_0+A_0(0)\right]D\delta^2\right).$$

2 Online Appendix B: Solving the Extended RB Model with Unknown Income Growth

In the extended model, we first guess that $J_t = -\alpha_0-\alpha_1 w_t-\alpha_2 y_t-\alpha_3 m_t$. The J function at time $t+\Delta t$ can thus be written as:[2]

$$\begin{aligned}J_{t+\Delta t} &\equiv J\left(w_{t+\Delta t}, y_{t+\Delta t}, m_{t+\Delta t}\right) = -\alpha_0-\alpha_1 w_{t+\Delta t}-\alpha_2 y_{t+\Delta t}-\alpha_3 m_{t+\Delta t} \\ &\approx -\alpha_0-\left[\alpha_1 w_t+\alpha_1\left(rw_t+y_t-c_t+\alpha_t\pi\right)\Delta t+\alpha_1\sigma_e\alpha_t\Delta B_{e,t}\right] \\ &-\left[\alpha_2 y_t+\alpha_2\left(\mu_2+\delta p_t-\rho y_t\right)\Delta t+\alpha_2\rho_{ye}\sigma_y\Delta B_{e,t}+\alpha_2\sqrt{1-\rho_{ye}^2}\sigma_y\Delta\hat{B}_{i,t}\right] \\ &-\left[\alpha_3 m_t+\alpha_3\lambda\left(\overline{\mu}-m_t\right)\Delta t+\alpha_3\sigma_m\Delta B_{m,t}\right].\end{aligned}$$

[2] Here $\Delta B_t = \sqrt{\Delta t}\epsilon$ and ϵ is a standard normal distributed variable.

Using the above expression for $J_{t+\Delta t}$ and assume that the time interval Δt goes to infinitesimal dt, we can compute the certainty equivalent of J_{t+dt} as follows:

$$\begin{aligned}
&\exp\left(-\gamma CE_t\right) = E_t\left[\exp\left(-\gamma J_{t+\Delta t}\right)\right] \\
&= \exp\left(-\gamma E_t\left[-\alpha_1 w_{t+dt} - \alpha_2 y_{t+dt} - \alpha_3 m_{t+dt}\right] + \frac{1}{2}\gamma^2 \operatorname{var}_t\left[-\alpha_1 w_{t+dt} - \alpha_2 y_{t+dt} - \alpha_3 m_{t+dt}\right] + \gamma\alpha_0\right) \\
&= \exp\left(\gamma\alpha_0 - \gamma\left(\partial J\right)^T \cdot \left(s_t + E\left[ds_t\right]\right) + \frac{\gamma^2}{2}\left[\left(\partial J\right)^T \cdot \Sigma \cdot \partial J\right] dt\right) \\
&= \exp\left(-\gamma J_t\right)\exp\left(-\gamma\left(\partial J\right)^T \cdot E\left[ds_t\right] + \frac{\gamma^2}{2}\left[\left(\partial J\right)^T \cdot \Sigma \cdot \partial J\right] dt\right), \qquad (8)
\end{aligned}$$

where $s_t = \begin{bmatrix} w_t & y_t & m_t \end{bmatrix}^T$, $ds_t = \begin{bmatrix} dw_t & dy_t & dm_t \end{bmatrix}^T$, $\partial J = \begin{bmatrix} J_w & J_y & J_m \end{bmatrix}^T$, and

$$\Sigma = \begin{bmatrix} \alpha_t^2\sigma_e^2 & \rho_{ey}\sigma_y\alpha_t\sigma_e & \rho_{ey}\alpha_t\sigma_e\sigma_m \\ \rho_{ey}\sigma_y\alpha_t\sigma_e & \sigma_y^2 & \sigma_y\sigma_m \\ \rho_{ey}\alpha_t\sigma_e\sigma_m & \sigma_y\sigma_m & \sigma_m^2 \end{bmatrix}.$$

Solving this equation yields:

$$CE_t\left[J_{t+dt}\right] = J_t + \left(\partial J \cdot E\left[ds_t\right] - \frac{\gamma}{2}\left[\partial J \cdot \Sigma \cdot \left(\partial J\right)^T\right] dt\right). \qquad (9)$$

Substituting the expression of CE_t into the HJB yields:

$$\delta V\left(J_t\right) = \max_{\{c_t,\alpha_t\}} \min_{\upsilon_t}\left\{\delta V\left(c_t\right) + \mathcal{D}V\left(s_t\right) + \frac{1}{2\vartheta_t}\left(\upsilon_t^T \cdot \Sigma \cdot \upsilon_t\right)\right\}, \qquad (10)$$

where

$$\mathcal{D}V\left(J_t\right) = V'\left(J_t\right)\left(\partial J \cdot E\left[ds_t\right] + \upsilon_t^T \cdot \Sigma \cdot \partial J - \frac{\gamma}{2}\left[\partial J \cdot \Sigma \cdot \left(\partial J\right)^T\right]\right).$$

Solving first for the infimization part of the robust HJB equation yields:

$$\upsilon_t = -\vartheta_t V'\left(J_t\right)\partial J \qquad (11)$$

Substituting this optimal distortion into the robust HJB equation yields:

$$\delta V\left(J_t\right) = \max_{\{c_t,\alpha_t\}}\left\{\delta V\left(c_t\right) + V'\left(J_t\right)\left(\partial J \cdot E\left[ds_t\right] - \frac{\gamma}{2}\left[\partial J \cdot \Sigma \cdot \left(\partial J\right)^T\right] - \frac{\vartheta_t}{2}V'\left(J_t\right)\left[\left(\partial J\right)^T \cdot \Sigma \cdot \partial J\right]\right)\right\}.$$

As in our benchmark model, we assume that $\vartheta_t = -\frac{\vartheta}{V(J_t)}$. Given that $V\left(J_t\right) = \left(-\psi\right)\exp\left(-J_t/\psi\right)$ and $V'\left(J_t\right) = \exp\left(-J_t/\psi\right)$, the HJB equation reduces to

$$\delta V\left(J_t\right) = \max_{\{c_t,\alpha_t\}}\left\{\delta V\left(c_t\right) + V'\left(J_t\right)\left(\partial J \cdot E\left[ds_t\right] - \frac{1}{2}\widetilde{\gamma}\left[\partial J \cdot \Sigma \cdot \left(\partial J\right)^T\right]\right)\right\},$$

where $\widetilde{\gamma} = \gamma + \vartheta/\psi$. The FOCs are

$$c_t = -\psi \ln\left(-\frac{\alpha_1}{\delta}\right) + \left(-\alpha_0 - \alpha_1 w_t - \alpha_2 y_t - \alpha_3 m_t\right), \tag{12}$$

and

$$\alpha_t = \frac{-\pi}{\widetilde{\gamma}\alpha_1\sigma_e^2} - \frac{\sigma_e\alpha_2\rho_{ye}\sigma_y}{\alpha_1\sigma_e^2} - \frac{\sigma_e\alpha_3\rho_{ey}\sigma_m}{\alpha_1\sigma_e^2}. \tag{13}$$

Substituting these expression back to the HJB yields:

$$\left(-\alpha_1 - \delta\right)\psi = \left(\begin{array}{c} \left[\begin{array}{c} -\alpha_1\left\{rw_t + y_t - \left[-\psi\ln\left(-\frac{\alpha_1}{\delta}\right) + \left(-\alpha_0 - \alpha_1 w_t - \alpha_2 y_t - \alpha_3 m_t\right)\right] + \alpha_t\pi\right\} \\ -\alpha_2\left(m_t - \rho y_t\right) - \alpha_3\lambda\left(\overline{\mu} - m_t\right) \end{array}\right] \\ -\frac{1}{2}\gamma\left\{\alpha_1^2\alpha_t^2\sigma_e^2 + \alpha_2^2\sigma_y^2 + \alpha_3^2\sigma_m^2 + 2\alpha_1\sigma_e\alpha_t\alpha_2\rho_{ye}\sigma_y + 2\alpha_1\sigma_e\alpha_t\alpha_3\rho_{ey}\sigma_m + 2\alpha_2\alpha_3\sigma_y\sigma_m\right\} \end{array}\right).$$

Matching the coefficients, we obtain:

$$\begin{aligned} \alpha_1 &= -r, \alpha_2 = -\frac{r}{r+\rho}, \alpha_3 = -\frac{r}{\left(r+\rho\right)\left(\lambda+r\right)}, \\ \alpha_0 &= \left(1 - \frac{\delta}{r}\right)\psi - \psi\ln\left(\frac{r}{\delta}\right) - \frac{\pi^2}{2r\widetilde{\gamma}\sigma_e^2} + \frac{\pi\rho_{ye}}{\sigma_e}\left[\frac{\sigma_y}{\left(r+\rho\right)} + \frac{\sigma_m}{\left(r+\rho\right)\left(\lambda+r\right)}\right] \\ &\quad - \frac{\lambda\overline{\mu}}{\left(r+\rho\right)\left(\lambda+r\right)} + \frac{1}{2}r\widetilde{\gamma}\left(1 - \rho_{ye}^2\right)\left[\frac{\sigma_y}{r+\rho} + \frac{\sigma_m}{\left(r+\rho\right)\left(\lambda+r\right)}\right]^2. \end{aligned}$$

Substituting these coefficients back to the FOCs, (12) and (13), yields the optimal consumption and portfolio rules under parameter and model uncertainty in the main text.

Substituting the consumption and portfolio rules into the expression for individual saving yields:

$$\begin{aligned} d_t^* &= ra_t + y_t - c_t^* + \pi\alpha^* \\ &= rw_t + y_t - r\left[\begin{array}{c} w_t + \frac{1}{r+\rho}\left(y_t + \frac{\overline{\mu}}{r} - \frac{\pi\rho_{ye}\sigma_y}{r\sigma_e}\right) \\ + \frac{1}{(r+\rho)(\lambda+r)}\left(m_t - \overline{\mu} - \frac{\pi\rho_{ye}\sigma_m}{r\sigma_e}\right) \end{array}\right] - \Psi + \Gamma - \Pi \\ &= y_t - \frac{r}{r+\rho}\left(y_t + \frac{\overline{\mu}}{r} - \frac{\pi\rho_{ye}\sigma_y}{r\sigma_e}\right) - \frac{r}{\left(r+\rho\right)\left(\lambda+r\right)}\left(m_t - \overline{\mu} - \frac{\pi\rho_{ye}\sigma_m}{r\sigma_e}\right) - \Psi + \Gamma - \Pi \\ &= \frac{\rho\left(y_t - \overline{y}\right)}{r+\rho} - \frac{r\left(m_t - \overline{\mu}\right)}{\left(r+\rho\right)\left(\lambda+r\right)} + \frac{\pi\rho_{ye}\sigma_y}{\left(r+\rho\right)\sigma_e} + \frac{\pi\rho_{ye}\sigma_m}{\left(r+\rho\right)\left(\lambda+r\right)\sigma_e} - \Psi + \Gamma - \Pi \\ &= \frac{\rho\left(y_t - \overline{y}\right)}{r+\rho} - \frac{r\left(m_t - \overline{\mu}\right)}{\left(r+\rho\right)\left(\lambda+r\right)} - \Psi + \Gamma_0, \end{aligned}$$

where

$$\Gamma_0 = \frac{1}{2}r\widetilde{\gamma}\left[\frac{\sigma_y}{r+\rho} + \frac{\sigma_m}{\left(r+\rho\right)\left(\lambda+r\right)}\right]^2$$

is the precautionary saving demand when $\rho_{ye} = 0$, and we use the facts that in equilibrium $\alpha^* = 0$, and

$$\pi^* = \frac{r\widetilde{\gamma}\rho_{ey}\sigma_e\sigma_y}{r+\rho} + \frac{r\widetilde{\gamma}\rho_{ey}\sigma_e\sigma_m}{\left(r+\rho\right)\left(r+\lambda\right)}.$$

References

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