Online Appendix for "Robustly Strategic Consumption-Portfolio Rules with Informational Frictions"

Yulei Luo*
The University of Hong Kong

Forthcoming in *Management Science*

1. Online Appendix A: The Equivalence between the Univariate and Multivariate RB Models

The original Caballero-Wang model has two state variables: w_t and y_t. The evil agent thus takes the transition dynamics of w_t and y_t,

$$dw_t = (rw_t + y_t - c_t)\,dt + \alpha_t\,(\pi dt + \sigma_e dB_{e,t}) \tag{1.1}$$

$$dy_t = \rho\left(\frac{\mu}{\rho} - y_t\right)dt + \sigma_y dB_t, \tag{1.2}$$

as the approximating model. Under RB, the HJB can be written as:

$$\sup_{c_t,\alpha_t}\inf_{\upsilon_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(w_t,y_t) + \mathcal{D}J(w_t,y_t) + \upsilon_t^T\cdot\Phi\cdot\partial J + \frac{1}{2\vartheta_t}\left(\upsilon_t^T\cdot\Phi\cdot\upsilon_t\right)\right], \tag{1.3}$$

where

$$\mathcal{D}J(w_t,y_t) = J_w(rw_t + y_t - c_t + \alpha_t\pi) + \frac{1}{2}J_{ww}\sigma_e^2\alpha_t^2 + J_y(\mu - \rho y_t) + \frac{1}{2}J_{yy}\sigma_y^2 + \rho_{ey}J_{yw}\sigma_y\sigma_e\alpha_t,$$

the fourth term, $\upsilon_t^T\cdot\Phi\cdot\partial J$, is the adjustment to the expected continuation value when the state dynamics is governed by the distorted model with the mean distortion υ_t, the final term, $\frac{1}{2\vartheta_t}\left(\upsilon_t^T\cdot\Phi\cdot\upsilon_t\right)$, quantifies the penalty due to RB, $\upsilon_t = \begin{bmatrix}\upsilon_{1,t}\\ \upsilon_{2,t}\end{bmatrix}$, $\Phi = \begin{bmatrix}\sigma_y^2 & \rho_{ey}\sigma_y\sigma_e\alpha_t\\ \rho_{ey}\sigma_y\sigma_e\alpha_t & \sigma_e^2\alpha_t^2\end{bmatrix}$, and $\partial J = \begin{bmatrix}J_y\\ J_w\end{bmatrix}$. Finally, the TVC,

$$\lim_{t\to\infty}E\left[\exp(-\delta t)\,|J(w_t,y_t)|\right] = 0,$$

holds.

*Faculty of Business and Economics, University of Hong Kong, Hong Kong. Email address: yulei.luo@gmail.com.

Solving first for the infimization part of the robust HJB equation (1.3) yields:

$$\sigma_y^2 J_y + \rho_{ey}\sigma_y\sigma_e\alpha_t J_w + \frac{1}{\vartheta_t}\left(\sigma_y^2 \upsilon_{1,t} + \rho_{ey}\sigma_y\sigma_e\alpha_t\upsilon_{2,t}\right) = 0,$$
$$\rho_{ey}\sigma_y\sigma_e\alpha_t J_y + \sigma_e^2\alpha_t^2 J_w + \frac{1}{\vartheta_t}\left(\rho_{ey}\sigma_y\sigma_e\alpha_t\upsilon_{1,t} + \sigma_e^2\alpha_t^2\upsilon_{2,t}\right) = 0,$$

which can be solved for υ_t:

$$\upsilon_t = \begin{bmatrix} \upsilon_{1,t} \\ \upsilon_{2,t} \end{bmatrix} = -\vartheta_t \begin{bmatrix} J_y \\ J_w \end{bmatrix}. \tag{1.4}$$

Substituting this worst-possible distortion back into (1.3) yields:

$$0 = \sup_{c_t,\alpha_t} \begin{bmatrix} -\frac{1}{\gamma}\exp\left(-\gamma c_t\right) - \delta J\left(w_t, y_t\right) + J_w\left(r w_t + y_t - c_t + \pi\alpha_t\right) + J_y\rho\left(\frac{\mu}{\rho} - y_t\right) \\ +\frac{1}{2}\left(J_{yy} - \vartheta_t J_y^2\right)\sigma_y^2 + \frac{1}{2}\left(J_{ww} - \vartheta_t J_w^2\right)\sigma_e^2\alpha_t^2 + \rho_{ey}\left(J_{yw} - \vartheta_t J_y J_w\right)\sigma_y\sigma_e\alpha_t. \end{bmatrix} \tag{1.5}$$

because

$$\upsilon_t^T \cdot \Phi \cdot \partial J + \frac{1}{2\vartheta_t}\left(\upsilon_t^T \cdot \Phi \cdot \upsilon_t\right) = -\frac{1}{2}\vartheta_t\left(\sigma_y^2 J_y^2 + \sigma_e^2\alpha_t^2 J_w^2 + 2\rho_{ey}\sigma_y\sigma_e\alpha_t J_y J_w\right).$$

Performing the indicated optimization yields the first-order conditions for c_t and α_t are:

$$c_t = -\frac{1}{\gamma}\ln\left(J_w\right), \tag{1.6}$$
$$\alpha_t = -\frac{J_w\pi}{\left(J_{ww} - \vartheta_t J_w^2\right)\sigma_e^2} - \frac{\rho_{ey}\sigma_e\left(J_{yw} - \vartheta_t J_y J_w\right)\sigma_y}{\left(J_{ww} - \vartheta_t J_w^2\right)\sigma_e^2} \tag{1.7}$$

respectively. Conjecture that the value function under RB is of the form:

$$J\left(w_t, y_t\right) = -\frac{1}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1 w_t - \alpha_2 y_t\right),$$

where α_0, α_1, and α_2 are constants to be determined. Using this guessed value function, we obtain:

$$J_y = \frac{\alpha_2}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1 w_t - \alpha_2 y_t\right), J_w = \exp\left(-\alpha_0 - \alpha_1 w_t - \alpha_2 y_t\right),$$
$$J_{ww} = -\alpha_1\exp\left(-\alpha_0 - \alpha_1 w_t - \alpha_2 y_t\right), J_{yy} = -\frac{\alpha_2^2}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1 w_t - \alpha_2 y_t\right)$$
$$J_{yw} = -\alpha_2\exp\left(-\alpha_0 - \alpha_1 w_t - \alpha_2 y_t\right).$$

In addition, like the univariate case, here we also assume that $\vartheta\left(w_t, y_t\right) = -\frac{\vartheta}{J(w_t,y_t)} = \frac{\alpha_1\vartheta}{\exp(-\alpha_0-\alpha_1 w_t-\alpha_2 y_t)} >$

0. Substituting these expressions into (1.5) yields: .

$$0 = \left[\begin{array}{c} -\frac{1}{\gamma} + \frac{\delta}{\alpha_1} + \left[r w_t + y_t - \frac{1}{\gamma}\left(\alpha_0 + \alpha_1 w_t + \alpha_2 y_t\right) + \pi \left(\frac{\pi}{\alpha_1 (1+\vartheta)\sigma_e^2} - \frac{\rho_{ey}\sigma_e \alpha_2 \sigma_y}{\alpha_1 \sigma_e^2} \right) \right] \\ + \frac{\alpha_2}{\alpha_1} \rho \left(\frac{\mu}{\rho} - y_t \right) - \frac{1}{2} \frac{\alpha_2^2}{\alpha_1} \left(1+\vartheta\right) \sigma_y^2 \\ -\frac{1}{2}\alpha_1 \left(1+\vartheta\right) \sigma_e^2 \left(\frac{\pi}{\alpha_1 (1+\vartheta)\sigma_e^2} - \frac{\rho_{ey}\sigma_e \alpha_2 \sigma_y}{\alpha_1 \sigma_e^2} \right)^2 - \alpha_2 \rho_{ey} \left(1+\vartheta\right) \sigma_y \sigma_e \left(\frac{\pi}{\alpha_1 (1+\vartheta)\sigma_e^2} - \frac{\rho_{ey}\sigma_e \alpha_2 \sigma_y}{\alpha_1 \sigma_e^2} \right). \end{array} \right], \tag{1.8}$$

where we use the facts that

$$c_t = \frac{1}{\gamma}\left(\alpha_0 + \alpha_1 w_t + \alpha_2 y_t\right),$$

$$\alpha = \frac{\pi}{\alpha_1 \left(1+\vartheta\right) \sigma_e^2} - \frac{\rho_{ey}\sigma_e \alpha_2 \sigma_y}{\alpha_1 \sigma_e^2}.$$

Matching the w_t and y_t terms yields:

$$\alpha_1 = r\gamma \text{ and } \alpha_2 = \frac{r\gamma}{r+\rho},$$

which means that $\alpha = \frac{\pi}{r\gamma(1+\vartheta)\sigma_e^2} - \frac{\rho_{ey}\sigma_y\sigma_e}{(r+\rho)\sigma_e^2}$ and it is the same portfolio rule as that obtained in the univariate case in the main text, $\alpha = \frac{\pi}{r\gamma(1+\vartheta)\sigma_e^2} - \frac{\rho_{ey}\sigma_e\sigma_s}{\sigma_e^2}$. Finally, matching the constant terms yields:

$$\begin{aligned} \alpha_0 &= \frac{\delta - r}{r} + \pi\alpha + \frac{\mu\gamma}{r+\rho} - \frac{1}{2}\frac{r\gamma^2}{(r+\rho)^2}\left(1+\vartheta\right)\sigma_y^2 - \frac{1}{2} r\gamma^2 \left(1+\vartheta\right)\gamma\sigma_e^2\alpha^2 - \rho_{ey}\left(1+\vartheta\right) r\gamma^2 \sigma_e \frac{\sigma_y}{r+\rho}\alpha \\ &= \frac{\delta - r}{r} + \pi\alpha + \frac{\mu\gamma}{r+\rho} - \frac{1}{2} r\gamma^2 \left(1+\vartheta\right) \left[\frac{\sigma_y^2}{(r+\rho)^2} + \sigma_e^2\alpha^2 + 2\rho_{ey}\sigma_e \frac{\sigma_y}{r+\rho}\alpha \right] \end{aligned}$$

which is just the same as that obtained in the univariate case,

$$\alpha_0 = \frac{\delta - r}{r} + \pi\alpha - \frac{1}{2} r\gamma^2 \left(1+\vartheta\right)\left(\sigma_s^2 + \sigma_e^2\alpha^2 + 2\rho_{ey}\sigma_e\sigma_s\alpha\right)$$

because $\sigma_s = \frac{\sigma_y}{r+\rho}$.

2. Online Appendix B: The Equivalence between Rational Inattention and Signal Extraction with Exogenous Noises

Dividing Λ on both sides of $\dot{\Sigma}_t = -\Lambda K_t^2 + 2r\Sigma_t + \sigma^2$, we obtain the following differential Riccati equation governing the evolution of K_t:

$$\dot{K}_t = -K_t^2 + 2rK_t + \frac{\sigma^2}{\Lambda}, \tag{2.1}$$

where σ^2/Λ is the signal-to-noise ratio (SNR) in this problem. In the steady state, we have the following proposition for this signal extraction case with exogenous noises:

Proposition 1. *Given* SNR (σ^2/Λ), *in the steady state, the evolution of the perceived state can be written as*

$$d\widehat{s}_t = (r\widehat{s}_t - c_t + \pi\alpha_t)\,dt + \widehat{\sigma}dB_t^*,$$

where

$$\widehat{\sigma} \equiv K\sqrt{\Lambda} = g(\tau)\,\sigma, \tag{2.2}$$

$$K = r + \sqrt{r^2 + \frac{\sigma^2}{\Lambda}}, \tag{2.3}$$

$g(\tau) \equiv r\sqrt{\tau} + \sqrt{1 + r^2\tau} > 1$, *and* $\tau \equiv 1/\,\mathrm{SNR} = \Lambda/\sigma^2$. *Furthermore, if* SNR *and* κ *satisfy the following equality:*

$$\mathrm{SNR} = 4\kappa(\kappa - r)\,,$$

then the RI and SE cases are observationally equivalent in the sense that they lead to the same model dynamics.

Proof. In the steady state in which $\dot{K}_t = 0$, solving the following algebraic Riccati equation,

$$-K_t^2 + 2rK_t + \frac{\sigma^2}{\Lambda} = 0,$$

yields the steady state Kalman gain:

$$K = r + \sqrt{r^2 + \frac{\sigma^2}{\Lambda}}. \tag{2.4}$$

and steady state conditional variance: $\Sigma = K\Lambda$. ■

3. Online Appendix C: Solving the Constraint Version of the RB Model

Following Hansen, Sargent, Turmuhambetova, and Williams (2006) and Hansen and Sargent (2007), we can have the following *constraint* specification of the above RB problem:

$$\sup_{c_t,\alpha_t}\inf_{\upsilon_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) + \upsilon(s_t)\,\sigma^2 J_s\right] \tag{3.1}$$

subject to

$$\frac{1}{2}\left(\upsilon(s_t)\,\sigma\right)^2 \leq \eta, \tag{3.2}$$

where $\mathcal{D}J(s_t) = J_s(rs_t - c_t + \pi\alpha_t) + \frac{1}{2}J_{ss}\left(\sigma_{et}^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t\right)$, and $\eta > 0$ measures the consumer's tolerance for model misspecification. It is clear from the above constraint that the worst-case distortion is

$$\upsilon^*(s_t) = -\sqrt{2\eta}/\sigma < 0.$$

Substituting for $\upsilon^* = -\sqrt{2\eta}/\sigma$ in the robust HJB equation gives:

$$\sup_{c_t,\alpha_t}\left[-\frac{1}{\gamma}\exp\left(-\gamma c_t\right) - \delta J\left(s_t\right) + J_s\left(rs_t - c_t + \pi\alpha_t - \sqrt{2\eta}\sigma\right) + \frac{1}{2}J_{ss}\sigma^2\right], \tag{3.3}$$

where $\sigma^2 = \sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t$. To solve this problem explicitly, here we consider two special cases: (1) $\rho_{ye} = \pm 1$ and (2) $\sigma_s = 0$.

3.1. Case 1: $\rho_{ye} = \pm 1$

In the first case, $\sigma = \sigma_e\alpha_t + \sigma_s$ when $\rho_{ye} = 1$.[1] Performing the indicated optimization yields the first-order conditions for c_t and α_t:

$$c_t = -\frac{1}{\gamma}\ln\left(J_s\right), \tag{3.4}$$

$$\alpha_t = \frac{\left(\pi - \sqrt{2\eta}\sigma_e\right)J_s + \sigma_s\sigma_e J_{ss}}{-J_{ss}\sigma_e^2}. \tag{3.5}$$

Substitute (3.4) and (3.5) back into (3.3) to arrive at the partial differential equation

$$0 = -\frac{J_s}{\gamma} - \delta J + \left[rs_t + \frac{1}{\gamma}\ln\left(J_s\right) + \left(\pi - \sqrt{2\eta}\sigma_e\right)\alpha_t - \sqrt{2\eta}\sigma_s\right]J_s + \frac{1}{2}J_{ss}\left(\sigma_e\alpha_t + \sigma_s\right)^2. \tag{3.6}$$

Conjecture that the value function is of the form $J\left(s_t\right) = -\frac{1}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1 s_t\right)$, where α_0 and α_1 are constants to be determined. Using this conjecture, we obtain that $J_s = \exp\left(-\alpha_0 - \alpha_1 s_t\right) > 0$ and $J_{ss} = -\alpha_1\exp\left(-\alpha_0 - \alpha_1 s_t\right) < 0$. (3.6) can thus be reduced to

$$-\delta\frac{1}{\alpha_1} = -\frac{1}{\gamma} + \left[rs_t - \left(\frac{\alpha_0}{\gamma} + \frac{\alpha_1}{\gamma}s_t\right) + \left(\pi - \sqrt{2\eta}\sigma_e\right)\frac{\left(\pi - \sqrt{2\eta}\sigma_e\right) - \alpha_1\sigma_s\sigma_e}{\alpha_1\sigma_e^2} - \sqrt{2\eta}\sigma_s\right] - \frac{1}{2}\alpha_1\left(\sigma_e\alpha_t + \sigma_s\right)^2.$$

Collecting terms, the undetermined coefficients in the value function turn out to be

$$\alpha_1 = r\gamma, \tag{3.7}$$

$$\alpha_0 = \frac{\delta - r}{r} + \frac{\left(\pi - \sqrt{2\eta}\sigma_e\right)\left[\left(\pi - \sqrt{2\eta}\sigma_e\right) - r\gamma\sigma_s\sigma_e\right]}{r\sigma_e^2} - \sqrt{2\eta}\gamma\sigma_s - \frac{1}{2}r\gamma^2\left(\sigma_e\alpha_t + \sigma_s\right)^2, \tag{3.8}$$

where $\alpha^* = \frac{\pi - \sqrt{2\eta}\sigma_e}{r\gamma\sigma_e^2} - \frac{\sigma_s}{\sigma_e}$. Substituting these back into the first-order condition (3.4) yields the consumption function

$$c_t^* = rs_t + \frac{\delta - r}{r\gamma} + \frac{\left(\pi - \sqrt{2\eta}\sigma_e\right)^2}{2r\gamma\sigma_e^2} - \frac{\pi\sigma_s}{\sigma_e}, \tag{3.9}$$

where we use the fact that $\sigma^2 = \sigma_e^2\alpha^{*2} + \sigma_s^2 + 2\sigma_s\sigma_e\alpha^* = \frac{\left(\pi - \sqrt{2\eta}\sigma_e\right)^2}{\left(r\gamma\sigma_e\right)^2}$.

[1] Note that when $\rho_{ye} = -1$, we can also solve the problem explicitly and our main conclusion still holds in this case.

Comparing with the consumption-portfolio rules obtained in the multiplier formulation of the RB model when $\rho_{ye} = 1$,

$$\alpha^* = \frac{\pi}{r\gamma\left(1+\vartheta\right)\sigma_e^2} - \frac{\sigma_s}{\sigma_e}, \tag{3.10}$$

$$c_t^* = rs_t + \frac{\delta - r}{r\gamma} + \frac{\pi^2}{2r\left(1+\vartheta\right)\gamma\sigma_e^2} - \frac{\pi\sigma_s}{\sigma_e}, \tag{3.11}$$

we can see that when

$$\eta = \frac{1}{2}\left(\frac{\vartheta}{1+\vartheta}\frac{\pi}{\sigma_e}\right)^2, \tag{3.12}$$

the multiplier and constraint formulations lead to the same portfolio rule α^*. In the multiplier formulation, the robustness parameter interacts with the CARA γ. Since they are multiplied by the variance σ_e^2, the risk is second order. In other words, the second order risk aversion is now enhanced by the presence of robustness measured by ϑ. In contrast, in the constraint formulation, the second order risk aversion is still governed by γ, and the robustness term, $\sqrt{2\eta}\sigma_e$, is proportional to the standard deviation σ_e. In other words, η measures the amount of the first order risk aversion.

Furthermore, it is straightforward to show that when this condition holds, the same amount of distortion is perceived under the two robustness formulations. That is,

$$\frac{\sqrt{2\eta}}{\sigma} = r\gamma\vartheta$$

holds when (3.12) holds.

3.2. Case 2: $\sigma_s = 0$

In this case, $\sigma = \sigma_e\alpha_t$ when $\sigma_s = 0$. Performing the indicated optimization yields the first-order conditions for c_t and α_t:

$$c_t = -\frac{1}{\gamma}\ln\left(J_s\right), \tag{3.13}$$

$$\alpha_t = \frac{\left(\pi - \sqrt{2\eta}\sigma_e\right)J_s}{-J_{ss}\sigma_e^2}. \tag{3.14}$$

Substitute (3.4) and (3.5) back into (3.3) to arrive at the partial differential equation

$$0 = -\frac{J_s}{\gamma} - \delta J + \left[rs_t + \frac{1}{\gamma}\ln\left(J_s\right) + \left(\pi - \sqrt{2\eta}\sigma_e\right)\alpha_t\right]J_s + \frac{1}{2}J_{ss}\left(\sigma_e\alpha_t\right)^2. \tag{3.15}$$

Conjecture that the value function is of the form $J\left(s_t\right) = -\frac{1}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1 s_t\right)$, where α_0 and α_1 are constants to be determined. Using this conjecture, we obtain that $J_s = \exp\left(-\alpha_0 - \alpha_1 s_t\right) > 0$ and

$J_{ss} = -\alpha_1 \exp(-\alpha_0 - \alpha_1 s_t) < 0$. (3.15) can thus be reduced to

$$-\delta \frac{1}{\alpha_1} = -\frac{1}{\gamma} + \left[r s_t - \left(\frac{\alpha_0}{\gamma} + \frac{\alpha_1}{\gamma} s_t \right) + \frac{\left(\pi - \sqrt{2\eta}\sigma_e\right)^2}{\alpha_1 \sigma_e^2} \right] - \frac{1}{2}\alpha_1 \left(\sigma_e \alpha_t\right)^2 .$$

Collecting terms, the undetermined coefficients in the value function turn out to be

$$\alpha_1 = r\gamma, \tag{3.16}$$

$$\alpha_0 = \frac{\delta - r}{r} + \frac{\left(\pi - \sqrt{2\eta}\sigma_e\right)^2}{r\sigma_e^2} - \frac{1}{2} r\gamma^2 \left(\sigma_e \alpha_t\right)^2 , \tag{3.17}$$

where $\alpha^* = \frac{\pi - \sqrt{2\eta}\sigma_e}{r\gamma\sigma_e^2}$. Substituting these back into the first-order condition (3.15) yields the consumption function

$$c_t^* = r s_t + \frac{\delta - r}{r\gamma} + \frac{\left(\pi - \sqrt{2\eta}\sigma_e\right)^2}{2r\gamma\sigma_e^2}, \tag{3.18}$$

where we use the fact that $\sigma^2 = \sigma_e^2 \alpha^{*2} = \frac{\left(\pi - \sqrt{2\eta}\sigma_e\right)^2}{\left(r\gamma\sigma_e\right)^2}$.

Comparing with the consumption-portfolio rules obtained in the multiplier formulation of the RB model when $\sigma_s = 0$,

$$\alpha^* = \frac{\pi}{r\gamma\left(1 + \vartheta\right)\sigma_e^2}, \tag{3.19}$$

$$c_t^* = r s_t + \frac{\delta - r}{r\gamma} + \frac{\pi^2}{2r\left(1 + \vartheta\right)\gamma\sigma_e^2}, \tag{3.20}$$

we can see that when

$$\eta = \frac{1}{2}\left(\frac{\vartheta}{1 + \vartheta} \frac{\pi}{\sigma_e} \right)^2, \tag{3.21}$$

the multiplier and constraint formulations lead to the same portfolio rule α^*. In the multiplier formulation, the robustness parameter interacts with the CARA γ. Since they are multiplied by the variance σ_e^2, the risk is second order. In contrast, in the constraint formulation, the second order risk aversion is still governed by γ, and the robustness term, $\sqrt{2\eta}\sigma_e$, is proportional to the standard deviation σ_e. In other words, η measures the amount of the first order risk aversion.

Furthermore, it is straightforward to show that when this condition holds, the same amount of distortion is perceived under these two robustness formulations. That is,

$$\frac{\sqrt{2\eta}}{\sigma} = r\gamma\vartheta$$

holds when (3.21) holds.

4. Online Appendix D: Solving the Multiple-Priors Model

Under ambiguity, the agent's beliefs are captured by not a single probability measure, but a set of probability measures equivalent to a reference probability measure. That is, the agent's belief can deviate from the reference probability measure within probability measures equivalent to it. We view the filtered model proposed in the paper which has the subjective probability measure $\mathbb{P}$ as the approximating (or reference) model and $\mathbb{P}$ as the reference probability measure. The approximating model serves as a benchmark among all the candidate models that an ambiguity-averse agent is willing to consider. The agent doubts that the approximating model is the true model governing the economy and only considers it as an approximation of the true model. He then considers a constrained set of alternative models that are sufficiently close to the approximating model. Therefore, the basic idea of the multiple-priors utility specification is the same as that of the Hansen-Sargent robust decision making specification.

As in Chen and Epstein (2002), the set of alternative models is constructed by density generators defined by θ_t. θ_t delivers a local distortion to the reference model. Let Θ be a set of density generators. For such a set Θ, define the set of probability measures, $\mathcal{P}$, which is generated by θ_t, by

$$\mathcal{P} = \left\{ \mathbb{P}^\theta \middle| \theta_t \in \Theta \right\},$$

where $\mathbb{P}^\theta$ is derived from $\mathbb{P}$. In order to analyze the agent's decision behavior under ambiguity in the framework of the recursive multiple priors utility defined by Chen and Epstein (2002), the agent's *priors* are assumed to be captured by the set of probability measures $\mathcal{P}$, and each density generator θ_t generates a martingale z_t^θ under the reference probability measure $\mathbb{P}$:

$$z_t^\theta = \exp\left\{ -\int_0^t \theta_s dB_t^* - \frac{1}{2}\int_0^t \theta_s^2 ds \right\}. \tag{4.1}$$

Then the set of priors $\mathcal{P}$ can be expressed by the Radon-Nikodym derivatives of the alternative probability measures with respect to the reference probability measure:

$$\mathcal{P} = \left\{ \mathbb{P}^\theta \middle| \theta_t \in \Theta, \frac{d\mathbb{P}^\theta}{d\mathbb{P}} = z_\infty^\theta \right\}. \tag{4.2}$$

Alternative probability measures are absolutely continuous with respect to the reference probability measure. If the set of priors $\mathcal{P}$ becomes large through the set of generators Θ, then the agent considers more situations including the worst and the best ones.

For any alternative probability measure $\mathbb{P}^\theta$, by Girsanov's theorem we can get

$$dB_t^\theta = dB_t^* + \theta_t dt,$$

where B_t^θ is a standard and independent Brownian motions under probability measure $\mathbb{P}^\theta$ and the filtration generated by B_t^θ coincides with $\{\mathcal{F}_t^y\}$. An ambiguity-averse agent is uncertain about whether

B_t^* is a Brownian motion with respect to his information filtration $\{\mathcal{F}_t^y\}$. Under the alternative probability measure $\mathbb{P}^\theta$, the distorted laws of the motions of s can be written as follows:

$$d\widehat{s}_t = (r\widehat{s}_t - c_t + \pi\alpha_t - \theta_t\widehat{\sigma})\, dt + \widehat{\sigma} d\widetilde{B}_t,$$

where the drift distortion is the product of the risks and the size of ambiguity. The larger the size of ambiguity is, the larger the magnitudes of the drift distortions, and the lower confidence in the reference model the agent has.

Let $J(\widehat{s}_t)$ denote the maximum feasible level of lifetime expected utility starting from time t:

$$J(\widehat{s}_t) = \max_{c_t,\alpha_t} \min_{\mathbb{P}^\theta\in\mathcal{P}} E^{\mathbb{P}^\theta}\left[\int_t^\infty u(c_s)e^{-\beta(s-t)}ds \middle| \mathcal{F}_t^Y\right],$$

subject to the state transition equations. The Hamilton-Jacobi-Bellman (HJB) equation for the agent's decision problem under ambiguity can be written as:

$$0 = \sup_{c_t,\alpha_t} \min_{\mathbb{P}^\theta\in\mathcal{P}} \left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(\widehat{s}_t) + (r\widehat{s}_t - c_t + \pi\alpha_t - \theta_t\widehat{\sigma}) J_{\widehat{s}} + \frac{1}{2}\widehat{\sigma}^2 J_{\widehat{s}\widehat{s}}\right]. \tag{4.3}$$

The optimal density generator is determined by:

$$\theta^* = \arg\max_{\theta\in\Theta}\{\theta_t\widehat{\sigma}J_{\widehat{s}}\} \tag{4.4}$$

For simplicity, following Chen and Epstein (2002), we consider a special type of ambiguity: a one-dimensional *κ-ignorance*. Then we define Θ as: $\Theta = \{\theta_t|\, \theta_t \in [-\theta,\theta], \kappa \geq 0\}$. The nonnegative constant θ represents the degree of ambiguity, that is, a large value of θ means larger degree of ambiguity. In other words, a larger value of θ implies the agent considers more scenarios including the best and the worst cases. Given that $J_{\widehat{s}} > 0$, (4.4) implies $\theta^* = \theta$. The HJB reduces to:

$$0 = \sup_{c_t,\alpha_t} \left[-\frac{1}{\gamma}\exp(-\gamma c_t) - \delta J(\widehat{s}_t) + (r\widehat{s}_t - c_t + \pi\alpha_t - \theta\widehat{\sigma}) J_{\widehat{s}} + \frac{1}{2}\widehat{\sigma}^2 J_{\widehat{s}\widehat{s}}\right], \tag{4.5}$$

where $\widehat{\sigma} \equiv f(\kappa)\sigma$ and $\sigma = \sqrt{\sigma_e^2\alpha^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha}$. Performing the indicated optimization yields the first-order conditions for c_t and α_t:

$$c_t = -\frac{1}{\gamma}\ln(J_{\widehat{s}}), \tag{4.6}$$

$$\alpha_t = \frac{(\pi/(f(\kappa)\sigma_e) - \theta) J_{\widehat{s}}}{-J_{\widehat{s}\widehat{s}} f(\kappa)\sigma_e} - \frac{\sigma_s}{\sigma_e}. \tag{4.7}$$

Note that here we set $\rho_{ye} = 1$ to solve the model explicitly. (Note that setting $\rho_{ye} = -1$ or $\sigma_s = 0$ can also help solve the model explicitly.)

Substitute the FOCs back into (4.5) to arrive at the following PDE:

$$0 = -\frac{J_{\widehat{s}}}{\gamma} - \delta J\left(\widehat{s}_t\right) + \left(r\widehat{s}_t + \frac{1}{\gamma}\ln\left(J_{\widehat{s}}\right) + \pi\alpha_t - \theta\widehat{\sigma}\right) J_{\widehat{s}} + \frac{1}{2}\widehat{\sigma}^2 J_{\widehat{s}\widehat{s}},$$

Conjecture that the value function is of the form: $J\left(\widehat{s}_t\right) = -\frac{1}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1\widehat{s}_t\right)$, where α_0 and α_1 are constants to be determined. Using this conjecture, we obtain $J_{\widehat{s}} = \exp\left(-\alpha_0 - \alpha_1\widehat{s}_t\right) > 0$ and $J_{\widehat{s}\widehat{s}} = -\alpha_1\exp\left(-\alpha_0 - \alpha_1\widehat{s}_t\right) < 0$. Substituting these expressions into the HJB yields:

$$0 = -\frac{1}{\gamma} + \delta\frac{1}{\alpha_1} + r\widehat{s}_t + \frac{1}{\gamma}\left(-\alpha_0 - \alpha_1\widehat{s}_t\right) + \pi\alpha - \theta\widehat{\sigma} - \frac{1}{2}\widehat{\sigma}^2\alpha_1,$$

Collecting terms, the undetermined coefficients in the value function turn out to be

$$\alpha_1 = r\gamma, \tag{4.8}$$

$$\alpha_0 = \frac{\delta - r}{r} + \pi\gamma\alpha - \theta\widehat{\sigma} - \frac{1}{2}r\gamma^2\widehat{\sigma}^2. \tag{4.9}$$

Substituting these coefficients into the FOCs yields:

$$\alpha_t = \frac{\pi/\left(f\left(\kappa\right)\sigma_e\right) - \theta}{r\gamma f\left(\kappa\right)\sigma_e} - \frac{\sigma_s}{\sigma_e},$$

$$c_t = r\widehat{s}_t + \frac{\delta - r}{\gamma r} + \pi\alpha - \frac{\theta}{\gamma}\widehat{\sigma} - \frac{1}{2}r\gamma\widehat{\sigma}^2$$

Comparing the results in this proposition with that in Proposition 3 of the paper, it is clear that rather than ϑ, θ measuring ambiguity enters the precautionary saving demand and the portfolio choice. More importantly, θ and ϑ, enter these functions in different ways. Specifically, if we compare the two expressions for the portfolio choice, precautionary saving demand, it is straightforward to show that the two specifications lead to the same portfolio choice rule when θ and ϑ satisfy the following equality:

$$\theta = \frac{\pi}{\sigma_e}\left(\frac{1}{f\left(\kappa\right)} - \frac{1}{1+\vartheta}\right). \tag{4.10}$$

5. Online Appendix E: Verifying the TVC in the RB–RI Model

Given that

$$J\left(\widehat{s}_t\right) = -\frac{1}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1\widehat{s}_t\right),$$

where

$$\alpha_1 = r\gamma, \tag{5.1}$$

$$\alpha_0 = \frac{\delta - r}{r} + \frac{\pi\left[\pi/f\left(\kappa\right) - \rho_{ye}\sigma_s\sigma_e r\alpha\left(1+\vartheta\right)\right]}{\left(1+\vartheta\right)r\sigma_e^2} - \frac{1}{2}rf\left(\kappa\right)^2\left(1+\vartheta\right)\gamma^2\left(\sigma_e^2\alpha_t + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t\right), \tag{5.2}$$

we can check if the investor's transversality condition (TVC), $\lim_{t\to\infty} E\left[\exp\left(-\delta t\right)\left|J\left(\widehat{s}_t\right)\right|\right] = 0$, is satisfied. Substituting the consumption-portfolio rules, c_t^* and α^*, into the state transition equation for $\widehat{s}_t$ yields:

$$d\widehat{s}_t = Adt + \widehat{\sigma} d\widetilde{B}_t,$$

where $A = -\frac{\delta - r}{r\gamma} + \frac{\pi^2}{2r\gamma\sigma_e^2} + \frac{1}{2}r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)\sigma_s^2$ under the approximating model. This Brownian motion with drift can be rewritten as:

$$s_t = s_0 + At + \widehat{\sigma}\left(\widetilde{B}_t - \widetilde{B}_0\right), \tag{5.3}$$

where $\widetilde{B}_t - \widetilde{B}_0 \sim N(0,t)$. Substituting (5.3) into $E\left[\exp\left(-\delta t\right)\left|J\left(\widehat{s}_t\right)\right|\right]$ yields:

$$\begin{aligned} E\left[\exp\left(-\delta t\right)\left|J\left(\widehat{s}_t\right)\right|\right] &= \frac{1}{\alpha_1}E\left[\exp\left(-\delta t - \alpha_0 - \alpha_1\widehat{s}_t\right)\right] \\ &= \frac{1}{\alpha_1}\exp\left(E\left[-\delta - \alpha_0 - \alpha_1\widehat{s}_t\right] + \frac{1}{2}\operatorname{var}\left(\alpha_1\widehat{s}_t\right)\right) \\ &= \frac{1}{\alpha_1}\exp\left(-\delta t - \alpha_0 - \alpha_1\left(s_0 + At\right) + \frac{1}{2}\alpha_1^2\widehat{\sigma}^2 t\right) \\ &= \left|J\left(\widehat{s}_0\right)\right|\exp\left(-\left(\delta + \alpha_1 A - \frac{1}{2}\alpha_1^2\widehat{\sigma}^2\right)t\right) \end{aligned}$$

where $\left|J\left(\widehat{s}_0\right)\right| = \frac{1}{\alpha_1}\exp\left(-\alpha_0 - \alpha_1\widehat{s}_0\right)$ is a positive constant and we use the facts that $\widehat{s}_t - \widehat{s}_0 \sim N\left(At, \widehat{\sigma}^2 t\right)$. Therefore, the TVC is satisfied if and only if the following condition holds:

$$\delta + \alpha_1 A - \frac{1}{2}\alpha_1^2\widehat{\sigma}^2 = r + \frac{\pi^2}{2\sigma_e^2} + \frac{1}{2}\left(r\gamma\right)^2\left[\left(1+\vartheta\right)\left(1-\rho_{ye}^2\right)\sigma_s^2 - f\left(\kappa\right)^2\left(\sigma_e^2\alpha_t^2 + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha_t\right)\right] > 0.$$

In the FI-RE case in which $\vartheta = 0$ and $\kappa = \infty$, this condition reduces to: $r + \pi^2/\left(2\sigma_e^2\right) - \left(r\gamma\right)^2\left(\rho_{ye}\sigma_s + \sigma_e\alpha^*\right)^2/2 > 0$. Using the parameter values we consider in the text, it is straightforward to show that the TVC is always satisfied in both the FI-RE and RB-RI models. It is straightforward to show that the TVC still holds under the distorted model in which $A = -\frac{\delta - r}{r\gamma} + \frac{\pi^2}{2r\gamma\sigma_e^2} + \frac{1}{2}r\widetilde{\gamma}\left(1-\rho_{ye}^2\right)\sigma_s^2 - r\gamma\vartheta\widehat{\sigma}^2$.

6. Online Appendix F: Optimality of Ex Post Gaussianity under RB

Following Sims (Section 5, 2003), we first define the expected loss function due to limited information-processing capacity as follows:

$$L_t = E_t\left[J_0\left(s_t\right) - J\left(x_t\right)\right], \tag{6.1}$$

where s_t is the unobservable state variable, x_t is the best estimate of the true state, $J\left(x_t\right) = -\exp\left(-\alpha_0 - \alpha_1 x_t\right)/\alpha_1$ is the value function under RI-RB, $J_0\left(s_t\right) = -\exp\left(-\beta_0 - \beta_1 s_t\right)/\beta_1$ is the corresponding value function when $\kappa = \infty$, $\alpha_1 = \beta_1$ and α_0 are given in (4.8) and (4.9), and $\alpha_0 = \beta_0$

when $\kappa = \infty$ and $f(\kappa) = 1$. It is straightforward to show that:

$$
\begin{aligned}
&\min E_t \left[J_0 (s_t) - J (x_t) \right] \\
&= \min E_t \left[-\frac{1}{\beta_1} \exp(-\beta_0 - \beta_1 s_t) + \frac{1}{\alpha_1} \exp(-\alpha_0 - \alpha_1 x_t) \right] \\
&\simeq \min E_t \left[\begin{array}{c} -\frac{1}{\beta_1} \exp(-\beta_0 - \beta_1 x_t) + \exp(-\beta_0 - \beta_1 x_t)(s_t - x_t) \\ -\frac{\beta_1}{2} \exp(-\beta_0 - \beta_1 x_t)(s_t - x_t)^2 + \frac{1}{\alpha_1} \exp(-\alpha_0 - \alpha_1 x_t) \end{array} \right] \\
&\Longleftrightarrow \min \left\{ -\frac{\beta_1}{2} \exp(-\beta_0 - \beta_1 x_t) E_t \left[(s_t - x_t)^2 \right] \right\} \\
&\Longleftrightarrow -\frac{\beta_1}{2} \exp(-\beta_0 - \beta_1 x_t) \min E_t \left[(s_t - x_t)^2 \right],
\end{aligned} \tag{6.2}
$$

where $x_t = E_t[s_t]$. The (approximate) loss function under RB-RI derived above is essentially the same as that obtained in the LQG RI model proposed in Sims (2003). Since the only difference in these two settings is just in the constant coefficient, β_0, in the loss function, RB does not affect the optimality of ex post Gaussianity in Sims' LQG setting after we approximate the value functions we obtained in the CARA-Gaussian setting.

7. Online Appendix G: Calibrating the Robustness Parameter in the RB-RI Model

The value of p is determined by the following procedure. Let model P denote the approximating model:

$$
d\widehat{s}_t = (r\widehat{s}_t - c_t + \pi\alpha_t) dt + \widehat{\sigma} d\widetilde{B}_t, \tag{7.1}
$$

and model Q be the distorted model:

$$
d\widehat{s}_t = (r\widehat{s}_t - c_t + \pi\alpha_t) dt + \widehat{\sigma} \left(\widehat{\sigma} v(\widehat{s}_t) dt + d\widetilde{B}_t \right). \tag{7.2}
$$

Define p_P as

$$
p_P = \text{Prob} \left(\ln \left(\frac{L_Q}{L_P} \right) > 0 \middle| P \right), \tag{7.3}
$$

where $\ln \left(\frac{L_Q}{L_P} \right)$ is the log-likelihood ratio. When model P generates the data, p_P measures the probability that a likelihood ratio test selects model Q. In this case, we call p_P the probability of the model detection error. Similarly, when model Q generates the data, we can define p_Q as

$$
p_Q = \text{Prob} \left(\ln \left(\frac{L_P}{L_Q} \right) > 0 \middle| Q \right). \tag{7.4}
$$

Given initial priors of 0.5 on each model and that the length of the sample is N, the detection error probability, p, can be written as:

$$
p(\vartheta; N) = \frac{1}{2} (p_P + p_Q), \tag{7.5}
$$

where ϑ is the robustness parameter used to generate model Q. Given this definition, we can see that $1-p$ measures the probability that econometricians can distinguish the approximating model from the distorted model.

The general idea of the calibration procedure is to find a value of ϑ such that $p(\vartheta;N)$ equals a given value (for example, 10%) after simulating model P, (7.1), and model Q, (7.2).[2] In the continuous-time model with the iid Gaussian specification, $p(\vartheta;N)$ can be easily computed. Because both models P and Q are arithmetic Brownian motions with constant drift and diffusion coefficients, the log-likelihood ratios are Brownian motions and are normally distributed random variables. Specifically, in the RB-RI model, let model P denote the approximating model, (7.1) and model Q be the distorted model, (7.2). Because both models P and Q are arithmetic Brownian motions with constant drift and diffusion coefficients under RB-RI, the log-likelihood ratios are normally distributed random variables. Consequently, the logarithm of the Radon-Nikodym derivative of the distorted model (Q) with respect to the approximating model (P) can be written as

$$\ln\left(\frac{L_Q}{L_P}\right) = -\int_0^N \overline{\upsilon} dB_s - \frac{1}{2}\int_0^N \overline{\upsilon}^2 ds, \tag{7.6}$$

where

$$\overline{\upsilon} \equiv \upsilon^* \widehat{\sigma} = -r\gamma\vartheta\sqrt{\sigma_e^2\alpha^{*2} + \sigma_s^2 + 2\rho_{ye}\sigma_s\sigma_e\alpha^*}. \tag{7.7}$$

Similarly, the logarithm of the Radon-Nikodym derivative of the approximating model (P) with respect to the distorted model (Q) is

$$\ln\left(\frac{L_P}{L_Q}\right) = \int_0^N \overline{\upsilon} dB_s + \frac{1}{2}\int_0^N \overline{\upsilon}^2 ds. \tag{7.8}$$

Given (7.6) and (7.8), it is straightforward to derive $p(\vartheta;N)$:

$$p(\vartheta;N) = \Pr\left(x < \frac{\overline{\upsilon}}{2}\sqrt{N}\right), \tag{7.9}$$

where x follows a standard normal distribution.

8. Online Appendix H: Comparison with Incomplete Information about Individual Income

In this section, we consider another widely-adopted type of informational frictions: incomplete information about the income process, and compare its implications for robustly strategic consumption-portfolio rules implications with that of RI we considered in the preceding section. Specifically, following Pischke (1995) and Wang (2004), we assume that where there are two individual components in the income process, agents can only observe the total income but have no way to distinguish

[2] The number of periods used in the calculation, N, is set to be the actual length of the data we study. For example, if we consider the post-war U.S. annual time series data provided by Robert Shiller from $1946-2010$, $T=65$.

the two individual components. Mathematically, we assume that labor income (y_t) has two distinct components ($y_{1,t}$ and $y_{2,t}$):

$$y_t = y_{1,t} + y_{2,t},$$

where

$$dy_{1,t} = (\mu_1 - \lambda_1 y_{1,t})\, dt + \sigma_1 dB_{1,t}, \tag{8.1}$$

$$dy_{2,t} = (\mu_2 - \lambda_2 y_{2,t})\, dt + \rho_{12}\sigma_2 dB_{1,t} + \sqrt{1-\rho_{12}}\sigma_2 dB_{2,t}, \tag{8.2}$$

and ρ_{12} is the instantaneous correlation between the two individual components, $y_{1,t}$ and $y_{2,t}$.[3] All the other notations are similar to what we used in our benchmark model. Without a loss of generality, we assume that $\lambda_1 < \lambda_2$ and $\sigma_1 > \sigma_2$. In other words, the first income component is more persistent and volatile than the second component. It is straightforward to show that if both components in the income process are observable, this model is essentially the same as our benchmark model with a univariate income process. In this incomplete-information case, we need to use the filtering technique to obtain the best estimates of the unobservable income components first and then solve the optimization problem given the estimated income components. Following the same technique adopted in Wang (2004), in the steady state in which the conditional variance-covariance matrix is constant, we can obtain the following updating equations for the conditional means of $(y_{1,t}, y_{2,t})$:

$$d\begin{pmatrix} \widehat{y}_{1,t} \\ \widehat{y}_{2,t} \end{pmatrix} = \begin{pmatrix} \mu_1 - \lambda_1 \widehat{y}_{1,t} \\ \mu_2 - \lambda_2 \widehat{y}_{2,t} \end{pmatrix} dt + \begin{pmatrix} \widehat{\sigma}_1 \\ \widehat{\sigma}_2 \end{pmatrix} dZ_t, \tag{8.3}$$

where $\widehat{y}_{i,t} = E_t[y_{i,t}]$ for $i = 1, 2$, $dZ_t \equiv \{dy_t - [(\mu_1 + \mu_2) + (\lambda_2 - \lambda_1)\widehat{y}_{1,t} - \lambda_2 y_t]\, dt\}/\sigma$ is a "constructed" innovation process using total income and the perceived individual components; Z_t is a standard Brownian motion; and $\sigma = \sqrt{\sigma_1^2 + 2\sigma_{12} + \sigma_2^2}$. $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are the standard deviations of $d\widehat{y}_{1,t}$ and $d\widehat{y}_{2,t}$ respectively:

$$\widehat{\sigma}_1 = \frac{1}{\sigma}\left[(\lambda_2 - \lambda_1)\Sigma_{11} + \sigma_1^2 + \sigma_{12}\right] \text{ and } \widehat{\sigma}_2 = \frac{1}{\sigma}\left[-(\lambda_2 - \lambda_1)\Sigma_{11} + \sigma_2^2 + \sigma_{12}\right],$$

where

$$\Sigma_{11} = \frac{1}{(\lambda_2 - \lambda_1)^2}\left(\sqrt{\Theta^2 + \sigma_1^2\sigma_2^2(\lambda_2 - \lambda_1)^2(1 - \rho_{12}^2)} - \Theta\right) \tag{8.4}$$

is the steady state conditional variance of $y_{1,t}$, $\Theta = \lambda_1\sigma_2^2 + \lambda_2\sigma_1^2 + (\lambda_1 + \lambda_2)\sigma_{12}$, and $\sigma_{12} = \rho_{12}\sigma_1\sigma_2$. It is worth noting that for this bi-variate Gaussian income specification, Σ_{11} can fully characterize the estimation risk induced by partially observed income.

Following the same procedure that I used in the preceding sections, we can solve this IC model with RB. The following proposition summarizes the solution to the above problem:

[3] Pischke (1995) considered a more special two-component income specification (one component is iid, and the other component is a random walk) in a discrete-time setting.

Proposition 1. *Given ϑ, the robust consumption-portfolio rules under IC are:*

$$c_t^* = r\left[w_t + \frac{1}{r+\lambda_1}\left(\widehat{y}_{1,t} + \frac{\mu_1}{r}\right) + \frac{1}{r+\lambda_2}\left(\widehat{y}_{2,t} + \frac{\mu_2}{r}\right)\right] \tag{8.5}$$
$$+\left[1 - \frac{1}{2}\left(\frac{1+\vartheta/(r\gamma)}{1+\vartheta}\right)\right]\frac{\pi^2}{r(1+\vartheta)\gamma\sigma_e^2} - \frac{\pi\rho_{ey}}{\sigma_e}\left(\frac{\widehat{\sigma}_1}{r+\lambda_1} + \frac{\widehat{\sigma}_2}{r+\lambda_2}\right) + \Psi - \Gamma,$$
$$\alpha^* = \frac{\pi}{r\gamma(1+\vartheta)\sigma_e^2} - \frac{\rho_{ey}\sigma_e}{\sigma_e^2}\left(\frac{\widehat{\sigma}_1}{r+\lambda_1} + \frac{\widehat{\sigma}_2}{r+\lambda_2}\right) \tag{8.6}$$

respectively, where $\Psi = (\delta - r)/(r\gamma)$ captures the dissavings effect of relative impatience, and

$$\Gamma = \frac{1}{2}r\widetilde{\gamma}\left(1-\rho_{ey}^2\right)\left(\frac{\widehat{\sigma}_1}{r+\lambda_1} + \frac{\widehat{\sigma}_2}{r+\lambda_2}\right)^2 \tag{8.7}$$

is the precautionary savings demand, where $\widetilde{\gamma} \equiv \gamma\left(1+\frac{\vartheta}{r\gamma}\right)$.

Proof. See Online Appendix I. ■

Comparing the robust portfolio rule obtained in the RB-RI model, $\alpha^* = \frac{\pi}{r\widetilde{\gamma}f(\kappa)\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}$, with (8.6), it is clear that RI and IC have distinct implications on robustly strategic asset allocation. Specifically, RI affects the speculation demand invested in the risky asset via $f(\kappa) = \sqrt{\kappa/(\kappa - r)} > 1$, whereas IC has no impact on the speculation demand (the first term in (8.6)). Furthermore, RI has no impact on the intertemporal hedging demand, whereas IC affects this demand via changing the volatility of perceived permanent income $\left(\frac{\widehat{\sigma}_1}{r+\lambda_1} + \frac{\widehat{\sigma}_2}{r+\lambda_2}\right)$. The main reason behind these results is that RI is applied to the state variable s that summarizes all of the relevant information in the state vector (w, y). In contrast, IC is only applied to the labor income process, and consumers under IC can observe financial wealth perfectly.

Furthermore, comparing the precautionary saving demand obtained in the RB-RI model, $\Gamma = \frac{1}{2}r\widetilde{\gamma}f(\kappa)^2\left(1-\rho_{ye}^2\right)\sigma_s^2$, with (8.7), we can see that RI and IC affect the precautionary saving demand via distinct channels. Specifically, RI increases precautionary savings by introducing the $f(\kappa)$ factor, whereas IC increases precautionary savings by increasing the variance of perceived permanent income from $\left(\frac{\sigma_1}{r+\lambda_1}\right)^2 + 2\frac{\rho_{12}\sigma_1\sigma_2}{(r+\lambda_1)(r+\lambda_2)} + \left(\frac{\sigma_2}{r+\lambda_2}\right)^2$ to $\left(\frac{\widehat{\sigma}_1}{r+\lambda_1} + \frac{\widehat{\sigma}_2}{r+\lambda_2}\right)^2$.

9. Online Appendix I: Solving the RB Model with Partially Observed Income

In the IC model, the perceived income process is governed by the following equation:

$$d\begin{pmatrix}\widehat{y}_{1,t}\\ \widehat{y}_{2,t}\end{pmatrix} = \begin{pmatrix}\mu_1 - \rho_1\widehat{y}_{1,t}\\ \mu_2 - \rho_2\widehat{y}_{2,t}\end{pmatrix}dt + \begin{pmatrix}\widehat{\sigma}_1\\ \widehat{\sigma}_2\end{pmatrix}dZ_t, \tag{9.1}$$

where $\widehat{y}_{i,t} = E_t[y_{i,t}]$ for $i = 1, 2$, $dZ_t \equiv \{dy_t - [(\mu_1+\mu_2) + (\rho_2 - \rho_1)\widehat{y}_{1,t} - \rho_2 y_t]\,dt\}/\sigma$ is a "constructed" innovation process using total income and the perceived individual components, Z_t is a standard Brownian motion, and $\sigma = \sqrt{\sigma_1^2 + 2\sigma_{12} + \sigma_2^2}$. $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are the standard deviations of

$d\widehat{y}_{1,t}$ and $d\widehat{y}_{2,t}$ respectively:

$$\widehat{\sigma}_1 = \frac{1}{\sigma}\left[(\rho_2-\rho_1)\Sigma_{11}+\sigma_1^2+\sigma_{12}\right] \text{ and } \widehat{\sigma}_2 = \frac{1}{\sigma}\left[-(\rho_2-\rho_1)\Sigma_{11}+\sigma_2^2+\sigma_{12}\right],$$

where $\Sigma_{11} = \frac{1}{(\rho_2-\rho_1)^2}\left(\sqrt{\Theta^2+\sigma_1^2\sigma_2^2(\rho_2-\rho_1)^2\left(1-\rho_{12}^2\right)}-\Theta\right)$. Following the same procedure used in the benchmark model, we can introduce RB into this incomplete-information (IC) model by assuming that the agents take (9.1) plus the budget constraint,

$$dw_t = (rw_t+\widehat{y}_{1,t}+\widehat{y}_{2,t}-c_t)\,dt+\alpha_t(\pi dt+\sigma_e dB_{e,t}), \tag{9.2}$$

as the approximating model. The corresponding distorted model can thus be written as:

$$d\widehat{y}_{1,t} = (\mu_1-\rho_1\widehat{y}_{1,t})\,dt+\widehat{\sigma}_1(\widehat{\sigma}_1\upsilon_{1,t}dt+dZ_t), \tag{9.3}$$

$$d\widehat{y}_{2,t} = (\mu_2-\rho_2\widehat{y}_{1,t})\,dt+\widehat{\sigma}_2(\widehat{\sigma}_2\upsilon_{2,t}dt+dZ_t), \tag{9.4}$$

$$dw_t = (rw_t+\widehat{y}_{1,t}+\widehat{y}_{2,t}-c_t+\alpha_t\pi dt)\,dt+\widehat{\sigma}_e(\widehat{\sigma}_e\upsilon_{3,t}dt+dZ_t), \tag{9.5}$$

where $\widehat{\sigma}_e = \alpha_t\sigma_e$, and we denote $\upsilon_t \equiv \begin{bmatrix}\upsilon_{1,t} & \upsilon_{2,t} & \upsilon_{3,t}\end{bmatrix}^T$ the distortion vector chosen by the evil agent.

Under RB, the HJB can be written as:

$$\sup_{c_t,\alpha_t}\inf_{\upsilon_t}\left[-\frac{1}{\gamma}\exp(-\gamma c_t)-\delta J(w_t,\widehat{y}_{1,t},\widehat{y}_{2,t})+\mathcal{D}J(w_t,\widehat{y}_{1,t},\widehat{y}_{2,t})+\upsilon_t^T\cdot\Phi\cdot\partial J+\frac{1}{2\vartheta_t}\left(\upsilon_t^T\cdot\Phi\cdot\upsilon_t\right)\right], \tag{9.6}$$

where the fourth term is the adjustment to the expected continuation value when the state dynamics is governed by the distorted model, the final term quantifies the penalty due to RB,

$$\mathcal{D}J(w_t,\widehat{y}_{1,t},\widehat{y}_{2,t}) = J_w(rw_t+\widehat{y}_{1,t}+\widehat{y}_{2,t}-c_t+\alpha_t\pi)+\frac{1}{2}J_{ww}\sigma_e^2\alpha_t^2+J_{\widehat{y}_1}(\mu_1-\rho_1\widehat{y}_{1,t})+\frac{1}{2}J_{\widehat{y}_1\widehat{y}_1}\widehat{\sigma}_1^2$$
$$+J_{\widehat{y}_2}(\mu_2-\rho_2\widehat{y}_{2,t})+\frac{1}{2}J_{\widehat{y}_2\widehat{y}_2}\widehat{\sigma}_2^2+J_{\widehat{y}_1\widehat{y}_2}\widehat{\sigma}_1\widehat{\sigma}_2+\rho_{ey}J_{\widehat{y}_1 w}\widehat{\sigma}_1\sigma_e\alpha_t+\rho_{ey}J_{\widehat{y}_2 w}\widehat{\sigma}_2\sigma_e\alpha_t,$$

$\upsilon_t = \begin{bmatrix}\upsilon_{1,t}\\ \upsilon_{2,t}\\ \upsilon_{3,t}\end{bmatrix}$, $\Phi = \begin{bmatrix}\widehat{\sigma}_1^2 & \widehat{\sigma}_1\widehat{\sigma}_2 & \rho_{ey}\widehat{\sigma}_1\widehat{\sigma}_e\\ \widehat{\sigma}_1\widehat{\sigma}_2 & \widehat{\sigma}_2^2 & \rho_{ey}\widehat{\sigma}_2\widehat{\sigma}_e\\ \rho_{ey}\widehat{\sigma}_1\widehat{\sigma}_e & \rho_{ey}\widehat{\sigma}_2\widehat{\sigma}_e & \widehat{\sigma}_e^2\end{bmatrix}$, and $\partial J = \begin{bmatrix}J_{\widehat{y}_1}\\ J_{\widehat{y}_2}\\ J_w\end{bmatrix}$, subject to the distorted equations, (9.3), (9.4), and (9.5).[4] In addition, the TVC, $\lim_{t\to\infty}E\left[\exp(-\delta t)\left|J(w_t,\widehat{y}_{1,t},\widehat{y}_{2,t})\right|\right]=0$, holds.

Solving first for the infimization part of the problem yields:

$$\widehat{\sigma}_1 J_{\widehat{y}_1}+\widehat{\sigma}_2 J_{\widehat{y}_2}+\rho_{ey}\widehat{\sigma}_e J_w+\frac{1}{\vartheta_t}(\upsilon_{1,t}\widehat{\sigma}_1+\upsilon_{2,t}\widehat{\sigma}_2+\upsilon_{3,t}\rho_{ey}\widehat{\sigma}_e)=0,$$

[4] Note that $y_t = y_{1,t}+y_{2,t} = \widehat{y}_{1,t}+\widehat{y}_{2,t}$.

where $\widehat{\sigma}_e = \sigma_e \alpha_t$. Substituting this condition back into the above HJB yields:

$$0 = \sup_{c_t, \alpha_t} \left[\begin{array}{c} -\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(w_t, \widehat{y}_{1,t}, \widehat{y}_{2,t}) + J_w (r w_t + \widehat{y}_{1,t} + \widehat{y}_{2,t} - c_t + \alpha_t \pi) \\ + J_{\widehat{y}_1} (\mu_1 - \rho_1 \widehat{y}_{1,t}) + J_{\widehat{y}_2} (\mu_2 - \rho_2 \widehat{y}_{2,t}) \\ + \frac{1}{2} \left(J_{\widehat{y}_1 \widehat{y}_1} - \vartheta_t J_{\widehat{y}_1}^2 \right) \widehat{\sigma}_1^2 + \frac{1}{2} \left(J_{\widehat{y}_2 \widehat{y}_2} - \vartheta_t J_{\widehat{y}_2}^2 \right) \widehat{\sigma}_2^2 + \frac{1}{2} \left(J_{ww} - \vartheta_t J_w^2 \right) \widehat{\sigma}_e^2 \\ + \left(J_{\widehat{y}_1 \widehat{y}_2} - \vartheta_t J_{\widehat{y}_1} J_{\widehat{y}_2} \right) \widehat{\sigma}_1 \widehat{\sigma}_2 + \rho_{ey} \left(J_{\widehat{y}_1 w} - \vartheta_t J_{\widehat{y}_1} J_w \right) \widehat{\sigma}_1 \widehat{\sigma}_e + \rho_{ey} \left(J_{w \widehat{y}_2} - \vartheta_t J_w J_{\widehat{y}_2} \right) \widehat{\sigma}_e \widehat{\sigma}_2. \end{array} \right]. \tag{9.7}$$

Performing the indicated optimization yields the first-order conditions for c_t and α_t are:

$$c_t = -\frac{1}{\gamma} \ln (J_w), \tag{9.8}$$

$$\alpha_t = -\frac{J_w \pi}{(J_{ww} - \vartheta_t J_w^2) \sigma_e^2} - \frac{\rho_{ey} \sigma_e \left[\left(J_{\widehat{y}_1 w} - \vartheta_t J_{\widehat{y}_1} J_w \right) \widehat{\sigma}_1 + \left(J_{\widehat{y}_2 w} - \vartheta_t J_w J_{\widehat{y}_2} \right) \widehat{\sigma}_2 \right]}{(J_{ww} - \vartheta_t J_w^2) \sigma_e^2} \tag{9.9}$$

respectively.

In the next step, conjecture that the value function is of the form:

$$J(w_t, \widehat{y}_{1,t}, \widehat{y}_{2,t}) = -\frac{1}{\alpha_1} \exp(-\alpha_0 - \alpha_1 w_t - \alpha_2 \widehat{y}_{1,t} - \alpha_3 \widehat{y}_{2,t}),$$

where α_0, α_1, α_2, and α_3 are constants to be determined, and $\vartheta(w_t, \widehat{y}_{1,t}, \widehat{y}_{2,t}) = -\frac{\vartheta}{J(w_t, \widehat{y}_{1,t}, \widehat{y}_{2,t})}$. Substituting these expression together with (9.8) and (9.9) into (9.7), we have

$$0 = \left\{ \begin{array}{c} -\frac{1}{\gamma} + \delta \frac{1}{\alpha_1} + \left(r w_t + \widehat{y}_{1,t} + \widehat{y}_{2,t} + \frac{1}{\gamma} (-\alpha_0 - \alpha_1 w_t - \alpha_2 \widehat{y}_{1,t} - \alpha_3 \widehat{y}_{2,t}) + \pi \left[\frac{\pi}{\alpha_1 (1+\vartheta) \sigma_e^2} - \frac{\rho_{ey} \sigma_e (\alpha_2 \widehat{\sigma}_1 + \alpha_3 \widehat{\sigma}_2)}{\alpha_1 \sigma_e^2} \right] \right) \\ + \frac{\alpha_2}{\alpha_1} (\mu_1 - \rho_1 \widehat{y}_{1,t}) + \frac{\alpha_3}{\alpha_1} (\mu_2 - \rho_2 \widehat{y}_{2,t}) + \frac{1}{2} \left(-\frac{\alpha_2^2}{\alpha_1} - \vartheta \left(\frac{\alpha_2}{\alpha_1} \right)^2 \right) \widehat{\sigma}_1^2 + \frac{1}{2} \left(-\frac{\alpha_3^2}{\alpha_1} - \vartheta \left(\frac{\alpha_3}{\alpha_1} \right)^2 \right) \widehat{\sigma}_2^2 \\ + \frac{1}{2} (-\alpha_1 - \vartheta) \widehat{\sigma}_e^2 + \left(-\frac{\alpha_2 \alpha_3}{\alpha_1} - \vartheta \frac{\alpha_2}{\alpha_1} \frac{\alpha_3}{\alpha_1} \right) \widehat{\sigma}_1 \widehat{\sigma}_2 + \rho_{ey} \left(-\alpha_2 - \vartheta \frac{\alpha_2}{\alpha_1} \right) \widehat{\sigma}_1 \widehat{\sigma}_e + \rho_{ey} \left(-\alpha_3 - \vartheta \frac{\alpha_3}{\alpha_1} \right) \widehat{\sigma}_e \widehat{\sigma}_2 \end{array} \right\}.$$

Matching the coefficients in the w_t, $\widehat{y}_{1,t}$, and $\widehat{y}_{2,t}$ terms yields:

$$\alpha_1 = r\gamma, \alpha_2 = \frac{r\gamma}{r + \rho_1}, \alpha_3 = \frac{r\gamma}{r + \rho_2}.$$

Finally, matching the constant term yields:

$$\alpha_0 = \frac{\delta - r}{r} + \pi \left[\frac{\pi}{r(1+\vartheta)\sigma_e^2} - \frac{\rho_{ey} \sigma_e \gamma}{\sigma_e^2} \left(\frac{\widehat{\sigma}_1}{r + \rho_1} + \frac{\widehat{\sigma}_2}{r + \rho_2} \right) \right] + \gamma \left(\frac{\mu_1}{r + \rho_1} + \frac{\mu_2}{r + \rho_2} \right)$$
$$- \frac{1}{2} r \gamma^2 \left(1 + \frac{\vartheta}{r\gamma} \right) \left[\left(\frac{\widehat{\sigma}_1}{r + \rho_1} + \frac{\widehat{\sigma}_2}{r + \rho_2} \right)^2 + 2 \left(\frac{\widehat{\sigma}_1}{r + \rho_1} + \frac{\widehat{\sigma}_2}{r + \rho_2} \right) \rho_{ey} \alpha \sigma_e + \alpha^2 \sigma_e^2 \right].$$

where we use the facts that $\alpha = \frac{\pi}{\alpha_1 (1+\vartheta) \sigma_e^2} - \frac{\rho_{ey} \sigma_e (\alpha_2 \widehat{\sigma}_1 + \alpha_3 \widehat{\sigma}_2)}{\alpha_1 \sigma_e^2}$. Substituting these back into the first-order condition (9.8) and (9.9) yields robust consumption-portfolio rules in the text.

Finally, since RB does not change the MPC out of w_t, $\widehat{y}_{1,t}$, and $\widehat{y}_{2,t}$, and optimal portfolio rule is constant, it is straightforward to show that the TVC holds.