

# Online Appendix for “Ambiguity, Low Risk-Free Rates, and Consumption Inequality” (Not for Publication)

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# 1 Online Appendix A: Solving the Rational Expectations (RE) and Robustness (RB) Versions of the Recursive Utility Model

The RE optimizing problem can be written as:

$$f(J_t) = \max_{c_t} \left\{ \left(1 - e^{-\delta\Delta t}\right) f(c_t) + e^{-\delta\Delta t} f(CE_t[J_{t+\Delta t}]) \right\}, \quad (1)$$

where  $f(J_t)$  is the value function. An educated guess is that  $J_t = As_t + A_0$ . The  $J$  function at time  $t + \Delta t$  can thus be written as

$$J(s_{t+\Delta t}) = As_{t+\Delta t} + A_0 \approx As_t + A(rs_t - c_t)\Delta t + A\sigma_s\Delta B_t + A_0,$$

where  $\Delta s_t \equiv s_{t+\Delta t} - s_t$  and  $\Delta s_t \approx (rs_t - c_t)\Delta t + \sigma_s\Delta B_t$ . (Here  $\Delta B_t = \sqrt{\Delta t}\epsilon$  and  $\epsilon$  is a standard normal distributed variable.)

Using the definition of the certainty equivalent of  $J_{t+\Delta t}$ , we have

$$\begin{aligned} \exp(-\gamma CE_t) &= E_t[\exp(-\gamma J(s_{t+\Delta t}))] \\ &= \exp\left(-\gamma AE_t[s_{t+\Delta t}] + \frac{1}{2}\gamma^2 A^2 \text{var}_t[s_{t+\Delta t}] - \gamma A_0\right) \\ &= \exp\left(-\gamma A[s_t + (rs_t - c_t)\Delta t] + \frac{1}{2}\gamma^2 A^2 \sigma_s^2 \Delta t - \gamma A_0\right), \end{aligned}$$

which means that

$$CE_t = A \left[ s_t + \left( rs_t - c_t - \frac{1}{2}\gamma A \sigma_s^2 \right) \Delta t \right] + A_0. \quad (2)$$

Substituting these expressions back into (1) yields:

$$0 = \max_{c_t} \left\{ \delta f(c_t) \Delta t + f'(J_t) \left( A(rs_t - c_t) - \frac{1}{2}\gamma A^2 \sigma_s^2 \right) \Delta t - \delta \Delta t f(J_t) \right\}$$

where we use the facts that  $e^{-\delta\Delta t} = 1 - \delta\Delta t$ ,

$$J_{t+\Delta t} \approx J_t + J'_t(rs_t - c_t)\Delta t = J_t + A(rs_t - c_t)\Delta t,$$

and

$$f\left(J_t + A(rs_t - c_t)\Delta t + \frac{1}{2}A^2\sigma_s^2\Delta t\right) \approx f(J_t) + f'(J_t)\left(A(rs_t - c_t) - \frac{1}{2}\gamma A^2\sigma_s^2\right)\Delta t.$$

Dividing both sides by  $\Delta t$ , the Bellman equation can then be simplified as:

$$\delta f(J_t) = \max_{c_t} \left\{ \delta f(c_t) + f'(J_t) \left( A(rs_t - c_t) - \frac{1}{2}\gamma A^2\sigma_s^2 \right) \right\}. \quad (3)$$

The FOC for  $c$  is then

$$\delta f'(c_t) = f'(J_t) A,$$

which implies that

$$c_t = -\psi \ln \left( \frac{A}{\delta} \right) + (As_t + A_0). \quad (4)$$

Substituting this expression for  $c$  back to the Bellman equation and matching the coefficients, we have:

$$A = r$$

and

$$A_0 = \psi \left( \frac{\delta - r}{r} \right) + \psi \ln \left( \frac{r}{\delta} \right) - \frac{1}{2} \gamma r \sigma_s^2 - \frac{\vartheta}{2\psi} r \sigma_s^2.$$

Substituting these coefficients into (4) gives the consumption function

$$c_t = r s_t + \Psi(r) - \Gamma(r), \quad (5)$$

where

$$\Psi(r) \equiv \psi \left( \frac{\delta}{r} - 1 \right) \quad (6)$$

is the savings demand due to relative patience (if  $\delta < r$ , this term is negative and so savings rises) and

$$\Gamma(r) \equiv \frac{1}{2} r \gamma \sigma_s^2, \quad (7)$$

is the consumer's precautionary saving demand.

Following Hansen and Sargent (2007), Uppal and Wang (2003), and Maenhout (2004), we introduce robustness into the above otherwise standard model as follows:

$$0 = \max_{c_t} \min_{v_t} \left\{ \delta f(c_t) \Delta t + f'(J_t) \left( A(r s_t - c_t) - \frac{1}{2} \gamma A^2 \sigma_s^2 \right) \Delta t - \delta f(J_t) \Delta t + \frac{1}{2(\theta_t/\Delta t)} v_t^2 \sigma_s^2 \right\}$$

subject to the distorting equation,  $\Delta s_t \approx (r s_t - c_t) \Delta t + \sigma_s (\sigma_s v_t \Delta t + \Delta B_t)$ . It is worth noting that here following Hansen and Sargent (2011) and Kasa and Lei (2017), we scale the robustness parameter ( $\theta_t$ ) by the sampling interval ( $\Delta t$ ), effectively making the consumer have stronger preference for robustness (or more ambiguity averse) as the sampling interval shrinks.

Dividing both sides by  $\Delta t$ , the Bellman equation reduces to:

$$\delta f(J_t) = \max_{c_t} \min_{v_t} \left\{ \delta f(c_t) + f'(J_t) A \left( r s_t - c_t - \frac{1}{2} \gamma A \sigma_s^2 + b_t \sigma_s^2 \right) + \frac{1}{2\theta_t} v_t^2 \sigma_s^2 \right\}, \quad (8)$$

subject to (??). Solving first for the infimization part of the problem yields

$$b^*(s_t) = -\theta_t A f'(J_t).$$

Given that  $\theta_t > 0$ , the perturbation adds a negative drift term to the state transition equation because  $f'(J_t) > 0$ . Substituting it into the above HJB equation yields:

$$\delta f(J_t) = \max_c \left\{ \delta f(c_t) + f'(J_t) A \left( r s_t - c_t - \frac{1}{2} \gamma A \sigma_s^2 - \theta_t A f'(U_t) \sigma_s^2 \right) + \frac{1}{2\theta_t} (\theta_t A f'(J_t))^2 \sigma_s^2 \right\} \quad (9)$$

Following Uppal and Wang (2003) and Maenhout (2004), we assume that

$$\theta_t = -\frac{\vartheta}{f(U_t)}.$$

The HJB equation reduces to

$$\delta f(U_t) = \max_c \left\{ \delta f(c_t) + f'(J_t) A \left( r s_t - c_t - \frac{1}{2} \gamma A \sigma_s^2 + \frac{\vartheta}{f(U_t)} A f'(U_t) \sigma_s^2 \right) - \frac{\vartheta}{2f(J_t)} A^2 (f'(J_t))^2 \sigma_s^2 \right\}.$$

The FOC for  $c$  is then

$$\delta f'(c_t) = f'(J_t) A,$$

which implies that

$$c_t = -\psi \ln \left( \frac{A}{\delta} \right) + (A s_t + A_0). \quad (10)$$

Substituting this expression for  $c$  back to the Bellman equation and matching the coefficients, we have:

$$A = r \text{ and } A_0 = \psi \left( \frac{\delta}{r} - 1 \right) + \psi \ln \left( \frac{r}{\delta} \right) - \frac{1}{2} \gamma r \sigma_s^2 - \frac{\vartheta}{2\psi} r \sigma_s^2.$$

Substituting these coefficients into (10) gives the consumption function and the value function in the main text.

Finally, we check if the consumer's transversality condition (TVC),

$$\lim_{t \rightarrow \infty} E[\exp(-\delta t) |f(s_t)|] = 0, \quad (11)$$

is satisfied. Substituting the consumption function,  $c_t^*$ , into the state transition equation for  $s_t$  yields

$$ds_t = \tilde{A} dt + \sigma_s dB_t,$$

where  $\tilde{A} = -\frac{\psi(\delta-r)}{r} + \frac{1}{2} r \tilde{\gamma} \sigma_s^2$  under the approximating model. This Brownian motion with drift can be rewritten as

$$s_t = s_0 + \tilde{A} t + \sigma (B_t - B_0), \quad (12)$$

where  $B_t - B_0 \sim N(0, t)$ . Substituting (12) into  $E[\exp(-\delta t) | f(s_t) |]$  yields:

$$\begin{aligned} E[\exp(-\delta t) | f(s_t) |] &= \frac{1}{\alpha_1} E[\exp(-\delta t - \alpha_0 - \alpha_1 s_t)] \\ &= \frac{1}{\alpha_1} \exp\left(E[-\delta - \alpha_0 - \alpha_1 s_t] + \frac{1}{2} \text{var}(\alpha_1 s_t)\right) \\ &= \frac{1}{\alpha_1} \exp\left(-\delta t - \alpha_0 - \alpha_1 (s_0 + \tilde{A}t) + \frac{1}{2} \alpha_1^2 \sigma_s^2 t\right) \\ &= |J(s_0)| \exp\left(-\left(\delta + \alpha_1 \tilde{A} - \frac{1}{2} \alpha_1^2 \sigma_s^2\right) t\right) \end{aligned}$$

where  $|J(s_0)| = \frac{1}{\alpha_1} \exp(-\alpha_0 - \alpha_1 s_0)$  is a positive constant and we use the facts that  $s_t - s_0 \sim N(\tilde{A}t, \sigma_s^2 t)$ . Therefore, the TVC, (11), is satisfied if and only if the following condition holds:

$$\delta + \alpha_1 \tilde{A} - \frac{1}{2} \alpha_1^2 \sigma_s^2 = r + \frac{1}{2} \left(\frac{r}{\psi}\right)^2 \left(\frac{\gamma}{\psi} - 1 + \vartheta\right) \sigma_s^2 > 0.$$

Given the parameter values we consider in the text, it is obvious that the TVC is always satisfied in both the FI-RE and RB models. It is straightforward to show that the TVC still holds under the distorted model in which  $\tilde{A} = -\frac{\psi(\delta-r)}{r} + \frac{1}{2} r \tilde{\gamma} \sigma_s^2 - \frac{r}{\psi} \vartheta \sigma_s^2$  for plausible values of  $\vartheta$ .

## 2 Online Appendix B: Description of Data

This appendix describes the data we use to estimate the income process as well as the method we use to construct a panel of both household income and consumption for our empirical analysis.

We use micro data from the Panel Study of Income Dynamics (PSID). Our household sample selection closely follows that of Blundell, Pistaferri, and Preston (2008).<sup>1</sup> We exclude households in the PSID low-income and Latino samples. We exclude household incomes in years of family composition change, divorce or remarriage, and female headship. We also exclude incomes in years where the head or wife is under 30 or over 65, or is missing education, region, or income responses. We also exclude household incomes where non-financial income is less than \$1000, where year-over-year income change is greater than \$90,000, and where year-over-year consumption change is greater than \$50,000. Our final panel contains 7,220 unique households with 54,901 yearly income responses and 50,422 imputed nondurable consumption values.<sup>2</sup>

<sup>1</sup>They create a new panel series of consumption that combines information from PSID and CEX, focusing on the period when some of the largest changes in income inequality occurred.

<sup>2</sup>There are more household incomes than imputed consumption values because food consumption - the main input variable in Guvenen and Smith's nondurable demand function - is not reported in the PSID for the years 1987 and 1988. Dividing the total income responses by unique households yields an average of 7 - 8 years of responses per household. These years are not necessarily consecutive as our sample selection procedure allows households to be excluded in certain years but return to the sample if they later meet the criteria once again.

The PSID does not include enough consumption expenditure data to create full picture of household nondurable consumption. Such detailed expenditures are found, though, in the Consumer Expenditure Survey (CEX) from the Bureau of Labor Statistics. However, households in the CEX are only interviewed for four consecutive quarters and thus do not form a panel. To create a panel of consumption to match the PSID income measures, we use an estimated demand function for imputing nondurable consumption created by Guvenen and Smith (2014). Using an IV regression, they estimate a demand function for nondurable consumption that fits the detailed data in the CEX. The demand function uses demographic information and food consumption which can be found in both the CEX and PSID. Thus, we use this demand function of food consumption and demographic information (including age, family size, inflation measures, race, and education) to estimate nondurable consumption for PSID households, creating a consumption panel.

In order to estimate the income process, we narrow the sample period to the years 1980 – 1996, due to the PSID survey changing to a biennial schedule after 1996. To further restrict the sample to exclude households with dramatic year-over-year income and consumption changes, we eliminate household observations in years where either income or consumption has increased more than 200 percent or decreased more than 80 percent from the previous year.

### 3 Online Appendix C: Solving the RU-RB Model with a Risky Asset

The robust HJB equation for the RU-RB model with multiple financial assets can be written as:

$$\delta f(J_t) = \max_c \min_v \left\{ \delta f(c_t) + f'(U_t) A \left( r s_t - c_t + \pi \chi_t - \frac{1}{2} \gamma A \sigma^2 + b_t \sigma^2 \right) + \frac{1}{2\theta_t} b_t^2 \sigma^2 \right\}, \quad (13)$$

subject to the distorting equation. Solving first for the infimization part of the problem yields

$$v^*(s_t) = -\theta_t A f'(J_t).$$

Given that  $\theta_t > 0$ , the perturbation adds a negative drift term to the state transition equation because  $f'(J_t) > 0$ . Substituting it into the above HJB equation yields:

$$\delta f(J_t) = \max_{\{c_t, \alpha_t\}} \left\{ \delta f(c_t) + f'(U_t) A \left( r s_t - c_t + \pi \alpha_t - \frac{1}{2} \gamma A \sigma^2 - \theta_t A f'(U_t) \sigma^2 \right) + \frac{1}{2\theta_t} (\theta_t A f'(U_t))^2 \sigma^2 \right\}$$

Following Uppal and Wang (2003) and Maenhout (2004), we assume that  $\theta_t = -\vartheta/f(U_t)$ . The HJB equation reduces to

$$\delta f(U_t) = \max_{\{c_t, \alpha_t\}} \left\{ \delta f(c_t) + f'(U_t) A \left( r s_t - c_t + \pi \alpha_t - \frac{1}{2} \gamma A \sigma^2 + \frac{\vartheta f'(U_t)}{f(U_t)} A \sigma^2 \right) - \frac{\vartheta (f'(U_t))^2}{2f(J_t)} A^2 \sigma^2 \right\}.$$

Using the fact that  $f(U_t) = (-\psi) \exp(-U_t/\psi)$ , the HJB reduces to

$$\delta f(U_t) = \max_{\{c_t, \alpha_t\}} \left\{ \delta f(c_t) + f'(U_t) A \left( r s_t - c_t + \pi \alpha_t - \frac{1}{2} \left( \gamma + \frac{\vartheta}{\psi} \right) A \sigma^2 + \frac{\vartheta f'(U_t)}{f(U_t)} A \sigma^2 \right) \right\}.$$

The FOC for  $c_t$  is then

$$\delta f'(c_t) = f'(J_t) A,$$

which implies that

$$c_t = -\psi \ln \left( \frac{A}{\delta} \right) + (A s_t + A_0). \quad (14)$$

The FOC for  $\alpha_t$  is

$$\alpha_t = \frac{\pi}{r(\gamma + \vartheta/\psi)\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2}, \quad (15)$$

which is just (??). Substituting this expression for  $c$  back to the Bellman equation and matching the coefficients, we have:

$$A = r \quad (16)$$

and

$$A_0 = \psi \left( \frac{\delta}{r} - 1 \right) - \frac{1}{2} r \tilde{\gamma} (1 - \rho_{ye}^2) \sigma_s^2 + \frac{\pi^2}{2r\tilde{\gamma}\sigma_e^2} - \frac{\pi\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} + \psi \ln \left( \frac{r}{\delta} \right). \quad (17)$$

Substituting these coefficients into (14) gives the consumption function, (??) in the main text.

Given the optimal consumption-portfolio rules, the individual saving function can be written as

$$\begin{aligned} d_t^* &= r a_t + y_t - c_t^* + \pi \alpha^* \\ &= r a_t + y_t - \left( r s_t + \Psi - \Gamma - \frac{\pi \rho_{ye} \sigma_s \sigma_e}{\sigma_e^2} + \frac{\pi^2}{2r\tilde{\gamma}\sigma_e^2} \right) + \pi \left( \frac{\pi}{r\tilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} \right) \\ &= \left[ r \left( a_t + \frac{1}{r + \rho_1} y_t + \frac{\rho_1}{r(r + \rho_1)} \bar{y} \right) - r \left( \frac{1}{r + \rho_1} y_t + \frac{\rho_1}{r(r + \rho_1)} \bar{y} \right) \right] + y_t \\ &\quad - \left( r s_t + \Psi - \Gamma - \frac{\pi \rho_{ye} \sigma_s \sigma_e}{\sigma_e^2} + \frac{\pi^2}{2r\tilde{\gamma}\sigma_e^2} \right) + \pi \left( \frac{\pi}{r\tilde{\gamma}\sigma_e^2} - \frac{\rho_{ye}\sigma_s\sigma_e}{\sigma_e^2} \right) \\ &= \frac{\rho}{r + \rho} (y_t - \bar{y}) + \Gamma - \Psi + \Pi, \end{aligned}$$

where  $\Pi = \frac{\pi^2}{2r\tilde{\gamma}\sigma_e^2}$ .