Robustness, The Risk-Free Rate, and Consumption Volatility in General Equilibrium*

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Abstract

This paper provides a tractable continuous-time constant-absolute-risk averse (CARA)-Gaussian framework to theoretically and quantitatively explore how the preference for robustness (RB) affects the interest rate, the dynamics of consumption and income, and the welfare costs of model uncertainty in general equilibrium. We first show that the equilibrium in this Huggett-type heterogeneous-agent economy exists and is unique for plausible parameter values. We then show that RB significantly reduces the equilibrium interest rate and the relative volatility of consumption growth to income growth when the income process is stationary, and can provide an explanation for the low risk free rate in the U.S. economy. Furthermore, we find that the welfare costs of model uncertainty are nontrivial for plausibly estimated and calibrated parameter values. Finally, we show that extending the benchmark model to the recursive utility setting can make the model better explain the observed relative volatility of consumption to income.

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1 Introduction

Hansen and Sargent (1995) first formally introduced the preference for robustness (RB, a concern for model misspecification) into linear-quadratic-Gaussian (LQG) economic models.[1] In robust control problems, agents are concerned about the possibility that their true model is misspecified in a manner that is difficult to detect statistically; consequently, they make their optimal decisions as if the subjective distribution over shocks is chosen by an evil agent in order to minimize their expected lifetime utility.[2] As showed in Hansen, Sargent, and Tallarini (HST, 1999) and Luo and Young (2010), and Luo, Nie, and Young (2012), robustness models can produce precautionary savings even within the class of discrete-time LQG models, which leads to analytical simplicity. Specifically, using the explicit consumption-saving rules, they explored how RB affects consumption and saving decisions and found that the preference for robustness and the discount factor are observationally equivalent in the sense that they lead to the identical consumption and saving decisions within the discrete-time representative-agent LQG setting. However, if we consider problems outside the LQG setting (e.g., when the utility function is constant-absolute-risk-averse, i.e., CARA, or constant-relative-risk-averse, i.e., CRRA), RB-induced worst-case distributions are generally non-Gaussian, which greatly complicates the computational task.[3]

The permanent income hypothesis (PIH) of Friedman states that the individual consumer's optimal consumption is determined by permanent income that equals the annuity value of his total resources: the sum of (i) financial wealth and (ii) human wealth defined as the discounted present value of the current and expected future labor income using the exogenously given risk-free rate. Hall (1978) showed that when some restrictions are imposed (e.g., quadratic utility and the equality between the interest rate and the discount rate), the PIH emerges and changes in consumption are unpredictable. Consequently, the PIH consumer saves only when he anticipates that their future labor income will decline. This saving motive is called the demand for "savings for a rainy day". In contrast, Caballero (1990) examined a precautionary saving motive due to the interaction of risk aversion and unpredictable future income uncertainty when the consumer has CARA utility. The Caballero model leads to a constant precautionary savings demand and a constant dissavings term due to relative impatience. Wang (2003) showed in a Bewley-Caballero-Huggett equilibrium model that the precautionary saving demand and the impatience dissavings term cancel out in a general equilibrium and the PIH reemerges.

The main goal of this paper is to construct a tractable continuous-time CARA-Gaussian heterogenous-agent dynamic stochastic general equilibrium (DSGE) model to link the two research lines discussed

[1]See Hansen and Sargent (2007) for a textbook treatment on robustness.

[2]The solution to a robust decision-maker's problem can be regarded as the equilibrium of a max-min game between the decision-maker and the evil agent.

[3]See Chapter 1 of Hansen and Sargent (2007) for discussions on the computational difficulties in solving non-LQG RB models, and Bidder and Smith (2012) and Young (2012) for using numerical methods to compute the worst-case distributions.

above and explore how robustness affects the interest rate, the cross-sectional distributions of consumption and income, and welfare costs of model uncertainty in the presence of uninsurable labor income.[4] As the first contribution of this paper, we show that this continuous-time DSGE model featuring incomplete markets and the separation of risk aversion and robustness can be solved explicitly. Using the explicit consumption-saving rules, we find that risk aversion is more important than robustness in determining the precautionary savings demand.[5] In addition, we show that there exists at least one equilibrium interest rate in this heteogenous-agent DSGE model with RB, and the equilibrium is unique for plausible parameter values.

Second, using the explicit decision rules, we show that a general equilibrium under RB can be constructed in the vein of Bewley (1986) and Huggett (1993).[6] In the general equilibrium, we find that the interest rate decreases with the degree of RB. The intuition is that the stronger the preference for RB, the greater the amount of model uncertainty determined by the interaction of risk aversion, RB, and labor income uncertainty, and the less the interest rate. In addition, we show that the relative volatility of consumption growth to income growth is determined by the interaction of the equilibrium interest rate and the persistence coefficient of the income process. Specifically, this relative volatility decreases with RB when the income process is stationary.

Third, after calibrating the RB parameter using the detection error probabilities (DEP), we find that RB has significant impacts on the equilibrium interest rate and consumption volatility. In the U.S. economy the average real risk-free interest rate is only about 1.87 percent between 1981 and 2010, and is only about 1.37% if the sample is from 1981 to 2015.[7] The full-information rational expectations (FI-RE) model requires the coefficient of risk aversion parameter to be above 6 to match this rate.[8] In contrast, when consumers take into account model uncertainty, the model can generate a low equilibrium interest rate with much lower values of the coefficient of risk aversion.[9] In addition, we find that when income uncertainty increases, the relative volatility of consumption growth to income growth decreases with the degree of robustness via the general equilibrium interest

[4] See Cagetti, Hansen, Sargent, and Williams (2002), Anderson, Hansen, and Sargent (2003), Maenhout (2004), and Kasa (2006) for the applications of robustness in continuous-time models.

[5] Within the discrete-time LQG setting, Luo, Nie, and Young (2012) showed that although both RB and CARA preferences increase the precautionary savings demand via the intercept terms in the consumption functions, they have distinct implications for the marginal propensity to consume out of permanent income (MPC).

[6] Wang (2003) constructed a general equilibrium under full-information rational expectations (FI-RE) in the same Bewley-Huggett type model economy with the CARA utility. Angeletos and Calvet (2006) characterized a closed-form recursive general equilibrium in a neoclassical growth model with idiosyncratic production risk and incomplete markets.

[7] Here the numbers are computed when we use CPI to measure inflation. Using PCE leads to the similar results. See Table 1 for different measures of the risk-free rates.

[8] Note that since we set the mean income level to be 1, the coefficient of relative risk aversion (CRRA) evaluated at this level is equal to the coefficient of absolute risk aversion (CARA).

[9] Barillas, Hansen, and Sargent (2009) showed that most of the observed high market price of risk in the U.S. can be reinterpreted as a market price of model uncertainty and the risk-aversion parameter can thus be reinterpreted as measuring the representative agent's doubts about the model specification.

rate channel.[10] We also find that when the benchmark model generates the observed low risk-free rate, the model's predicted relative volatility of consumption to income is well below the empirical counterpart.

Fourth, using the Lucas elimination-of-risk method, we find that the welfare costs due to model uncertainty are non-trivial. For plausibly parameter values, they could be as high as 10 percent of the typical consumer's total wealth.

Finally, we assume that consumers have stochastic differential utility (SDU or recursive utility) and have distinct preferences for risk and intertemporal substitution. After solving the model explicitly, we explore how it interacts with RB and affects the equilibrium interest rate and consumption volatility, and show that introducing RU can simultaneously explain the observed low risk-free rate and higher relative volatility of consumption growth to income growth.

This paper is organized as follows. Section 2 presents a robustness version of the Caballero–Bewley-Huggett type model with incomplete markets and precautionary savings. Section 3 discusses the general equilibrium implications of RB for the interest rate and consumption and wealth dynamics. Section 4 presents our quantitative results after estimating the income process and calibrating the RB parameter. Section 5 examines the implications of RB for the welfare cost of volatility in this general equilibrium model. Section 6 considers the extension to the SDU. Section 7 concludes.

2 A Continuous-time Heterogeneous-Agent Economy with Robustness

2.1 The Full-information Rational Expectations Model with Precautionary Savings

Following Wang (2003, 2009), we first formulate a continuous-time full-information rational expectations (FI-RE) Caballero-type model with precautionary savings. Specifically, we assume that there is only one risk-free asset in the model economy and there are a continuum of consumers who face uninsurable labor income and make optimal consumption-saving decisions. Uninsurable labor income (y_t) is assumed to follow an Ornstein-Uhlenbeck process:

$$dy_t = \rho \left(\frac{\mu}{\rho} - y_t \right) dt + \sigma_y dB_t, \tag{1}$$

where the unconditional mean and variance of income are $\overline{y} = \mu/\rho$ and $\sigma_y^2/(2\rho)$, respectively, the persistence coefficient ρ governs the speed of convergence or divergence from the steady state,[11] B_t is a standard Brownian motion on the real line $\mathcal{R}$, and σ_y is the unconditional volatility of the

[10] This theoretical result might provide a potential explanation for the empirical evidence documented in Blundell, Pistaferri, and Preston (2008) that income and consumption inequality diverged over the sampling period they study.

[11] If $\rho > 0$, the income process is stationary and deviations of income from the steady state are temporary; if $\rho \leq 0$, income is non-stationary. The last case catches the flavor of Hall and Mishkin (1982)'s the specification of individual

income change over an incremental unit of time. The typical consumer is assumed to maximize the following expected lifetime utility:

$$J_0 = E_0 \left[\int_{t=0}^{\infty} \exp\left(-\delta t\right) u(c_t) dt \right], \tag{2}$$

subject to the evolution of financial wealth (w_t):

$$dw_t = \left(r w_t + y_t - c_t\right) dt, \tag{3}$$

where r is the return to the risk-free asset, c is consumption, and the utility function takes the CARA form: $u(c_t) = -\exp\left(-\gamma c_t\right)/\gamma$, where $\gamma > 0$ is the coefficient of absolute risk aversion.[12] To present the model more compact, we define a new state variable, s_t:

$$s_t \equiv w_t + h_t,$$

where h_t is human wealth at time t and is defined as the expected present value of current and future labor income discounted at the risk-free interest rate r:

$$h_t \equiv E_t \left[\int_{t}^{\infty} \exp\left(-r\left(s-t\right)\right) y_s ds \right].$$

For the given the income process, (1), $h_t = y_t / \left(r + \rho\right) + \mu / \left(r\left(r + \rho\right)\right)$.[13] Following the state-space-reduction approach proposed in Luo (2008) and using the new state variable s, we can rewrite (3) as

$$ds_t = \left(r s_t - c_t\right) dt + \sigma_s dB_t, \tag{4}$$

where $\sigma_s = \sigma_y / \left(r + \rho\right)$ is the unconditional variance of the innovation to s_t.[14] It is not difficult to show that the above model with the univariate income process, (1), can be easily extended to the model with distinguishable multiple income components that have differencing persistence and volatility coefficients. In this more complicated case, we can still apply the state-space-reduction approach to simplify the model. To make our benchmark model tractable, we focus on the univariate income specification.

In this benchmark full-information rational expectations (FI-RE) model, we assume that the consumer trusts the model and observes the state perfectly, i.e., no model uncertainty and no state

income that includes a non-stationary component. The $\rho = 0$ case corresponds to a simple Brownian motion without drift. The larger ρ is, the less y tends to drift away from $\overline{y}$. As ρ goes to ∞, the variance of y goes to 0, which means that y can never deviate from $\overline{y}$.

[12] It is well-known that the CARA utility specification is tractable for deriving optimal policies and constructing general equilibrium in different settings. See Caballero (1990), Wang (2003, 2009), and Angeletos and Calvet (2006).

[13] Here we need to impose the restriction that $r > -\rho$ to guarantee the finiteness of human wealth.

[14] In the next section, we will introduce robustness directly into this "reduced" precautionary savings model. It is not difficult to show that the reduced univariate model and the original multivariate model are equivalent in the sense that they lead to the same consumption and saving functions. The detailed proof is available from the corresponding author by request.

uncertainty. Denoting the value function by $J(s_t)$. The Hamilton-Jacobi-Bellman (HJB) equation for this optimizing problem can be written as:

$$0 = \sup_{c_t \in \mathcal{C}} \left[-\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) \right],$$

where

$$\mathcal{D}J(s_t) = J_s(rs_t - c_t) + \frac{1}{2} J_{ss} \sigma_s^2, \tag{5}$$

$\mathcal{C}$ is the set of admissible values for the consumption choice, and the transversality condition, $\lim_{t\to\infty} E\left|\exp(-\delta t) J_t\right| = 0$, hold at optimum. Solving the above HJB subject to (4) leads to the following consumption function:

$$c_t = rs_t + \Psi - \Gamma, \tag{6}$$

where $\Psi = (\delta - r)/(r\gamma)$ and

$$\Gamma \equiv \frac{1}{2} r\gamma\sigma_s^2, \tag{7}$$

is the consumer's precautionary saving demand. Following the literature of precautionary savings, we measure the demand for precautionary saving as the amount of saving due to the interaction of the degree of risk aversion and uninsurable labor income risk. From (7), it can see that the precautionary saving demand is larger for a larger value of the coefficient of absolute risk aversion (γ), a more volatile income innovation (σ_y), and a larger persistence coefficient (ρ).[15] It is worth noting that although incomplete markets generally imply that aggregate dynamics depend on the wealth distribution, this "curse of dimensionality" can be overcome by our CARA-Gaussian specification under which investment is independent of wealth.

2.2 Incorporating Model Uncertainty due to Robustness

Robustness (robust control or robust filtering) emerged in the engineering literature in the 1970s and was introduced into economics and further developed by Hansen, Sargent, and others. A simple version of robustness considers the question of how to make optimal decisions when the decision maker does not know the true probability model that generates the data. The main goal of introducing robustness is to design optimal policies that not only work well when the reference (or approximating) model governing the evolution of the state variables is the true model, but also perform reasonably well when there is some type of model misspecification. To introduce robustness into our model proposed above, we follow the continuous-time methodology proposed by Anderson, Hansen, and Sargent (2003) (henceforth, AHS) and adopted in Maenhout (2004) to assume that consumers are concerned about the model misspecifications and take Equation (4) as

[15] As argued in Caballero (1990) and Wang (2009), a more persistent income shock takes a longer time to wear off and thus induces a stronger precautionary saving demand of a prudent forward-looking consumer.

the approximating model.[16] The corresponding distorting model can thus be obtained by adding an endogenous distortion $\upsilon(s_t)$ to (4):

$$ds_t = (rs_t - c_t)\,dt + \sigma_s\left(\sigma_s\upsilon(s_t)\,dt + dB_t\right). \tag{8}$$

As shown in AHS (2003), the objective $\mathcal{D}J$ defined in (5) can be thought of as $E[dJ]/dt$ and plays a key role in introducing robustness. A key insight of AHS (2003) is that this differential expectations operator reflects a particular underlying model for the state variable because this operator is determined by the stochastic differential equations of the state variables. Consumers accept the approximating model, (4), as the best approximating model, but is still concerned that it is misspecified. They therefore want to consider a range of models (i.e., the distorted model, (8)) surrounding the approximating model when computing the continuation payoff. A preference for robustness is then achieved by having the agent guard against the distorting model that is reasonably close to the approximating model. The drift adjustment $\upsilon(s_t)$ is chosen to minimize the sum of (i) the expected continuation payoff adjusted to reflect the additional drift component in (8) and (ii) an entropy penalty:

$$\inf_{\upsilon}\left[\mathcal{D}J + \upsilon(s_t)\,\sigma_s^2 J_s + \frac{1}{\vartheta_t}\mathcal{H}\right], \tag{9}$$

where the first two terms are the expected continuation payoff when the state variable follows (8), i.e., the alternative model based on drift distortion $\upsilon(s_t)$, $\mathcal{H} = (\upsilon(s_t)\,\sigma_s)^2/2$ is the relative entropy or the expected log likelihood ratio between the distorted model and the approximating model and measures the distance between the two models, and $1/\vartheta_t$ is the weight on the entropy penalty term.[17] ϑ_t is fixed and state independent in AHS (2003), whereas it is state-dependent in Maenhout (2004). The key reason of using a state-dependent counterpart ϑ_t in Maenhout (2004) is to assure the homotheticity or scale invariance of the decision problem with the CRRA utility function.[18] Note that the evil agent's minimization problem, (9), becomes invariant to the scale of total resource s_t when using the state-dependent specification of ϑ_t. In this paper, we also specify that ϑ_t is state-dependent ($\vartheta(s_t)$) in the CARA-Gaussian setting. The main reason for this specification is to guarantee the homotheticity, which makes robustness not wear off as the value of the total wealth increases.[19]

[16] As argued in Hansen and Sargent (2007), the agent's committment technology is irrelevant under RB if the evolution of the state is backward-looking. We therefore do not specifiy the committment technology of the consumer in the RB models of this paper.

[17] The last term in (9) is due to the consumer's preference for robustness. Note that the $\vartheta_t = 0$ case corresponds to the standard expected utility case. This robustness specification is called the *multiplier (or penalty)* robust control problem. We will discuss another closely related robustness specification, the *constraint* robust control problem, in the next subsection. See AHS (2003) and Hansen, Sargent, Turmuhambetova, and Williams (2006) (henceforth, HSTW) for detailed discussions on these two robustness specifications.

[18] See Maenhout (2004) for detailed discussions on the appealing features of "homothetic robustness".

[19] In the detailed procedure of solving the robust HJB proposed in Appendix 8.2, it is clear that the impact of robustness wears off if we assume that ϑ_t is constant.

Applying these results in the above model yields the following HJB equation under robustness:

$$\sup_{c_t \in \mathcal{C}} \inf_{\upsilon_t} \left[-\frac{1}{\gamma} \exp\left(-\gamma c_t\right) - \delta J\left(s_t\right) + \mathcal{D}J\left(s_t\right) + \upsilon\left(s_t\right) \sigma_s^2 J_s + \frac{1}{\vartheta\left(s_t\right)} \mathcal{H} \right]. \tag{10}$$

Solving first for the infimization part of (10) yields:

$$\upsilon\left(s_t\right)^* = -\vartheta\left(s_t\right) J_s,$$

where $\vartheta\left(s_t\right) = -\vartheta / J\left(s_t\right) > 0$. (See Appendix 8.2 for the derivation.) Following Uppal and Wang (2003) and Liu, Pan, and Wang (2005), here we can also define "$1/J\left(s_t\right)$" in the $\vartheta\left(s_t\right)$ specification as a *normalization* factor that is introduced to convert the relative entropy (i.e., the distance between the approximating model and the distorted model) to units of utility so that it is consistent with the units of the expected future value function evaluated with the distorted model. It is worth noting that adopting a slightly more general specification, $\vartheta\left(s_t\right) = -\varphi\vartheta / J\left(s_t\right)$ where φ is a constant, does not affect the main results of the paper. The reason is as follows. We can just define a new constant, $\widetilde{\vartheta} = \varphi\vartheta$, and $\widetilde{\vartheta}$, rather than ϑ, will enter the decision rules. Using a given detection error probability, we can easily calibrate the corresponding value of $\widetilde{\vartheta}$ that affects the optimal consumption-portfolio rules.[20]

Since $\vartheta\left(s_t\right) > 0$, the perturbation adds a negative drift term to the state transition equation because $J_s > 0$. Substituting for υ^* in (10) gives:

$$\sup_{c_t \in \mathcal{C}} \left[-\frac{1}{\gamma} \exp\left(-\gamma c_t\right) - \delta J\left(s_t\right) + \left(r s_t - c_t\right) J_s + \frac{1}{2} \sigma_s^2 J_{ss} - \frac{1}{2} \vartheta\left(s_t\right) \sigma_s^2 J_s^2 \right]. \tag{11}$$

2.3 The Robust Consumption Function and Precautionary Saving

Following the standard procedure, we can then solve (11) and obtain the consumption rule under robustness. The following proposition summarizes the solution:

1 *Under robustness, the consumption function and the saving function are*

$$c_t^* = r s_t + \Psi - \Gamma, \tag{12}$$

and

$$d_t^* = f_t + \Gamma - \Psi, \tag{13}$$

respectively, where $f_t = \rho\left(y_t - \overline{y}\right) / \left(r + \rho\right)$ *is the demand for savings "for a rainy day",* $\Psi\left(r\right) = \left(\delta - r\right) / \left(r\gamma\right)$ *captures the dissavings effect of relative impatience,*

$$\Gamma \equiv \frac{1}{2} r \widetilde{\gamma} \sigma_s^2 \tag{14}$$

[20] See Section 4.2 for the detailed procedure to calibrate the value of ϑ using the detection error probabilities.

is the demand for precautionary savings due to the interaction of income uncertainty, risk aversion, and uncertainty aversion, and $\widetilde{\gamma} \equiv (1+\vartheta)\,\gamma$ *is the effective coefficient of absolute risk aversion. Finally, the worst possible distortion is*

$$\upsilon^* = -r\gamma\vartheta. \tag{15}$$

Proof. See Appendix 8.2. ■

From (12), it is clear that robustness does not change the marginal propensity to consume out of permanent income (MPC), but affects the amount of precautionary savings (Γ). In other words, in the continuous-time setting, consumption is less sensitive to unanticipated income shocks than that predicted in the discrete-time robust LQG-PIH model of Hansen, Sargent, and Tallarini (1999) (henceforth, HST). In HST (1999), the MPC increases with model uncertainty determined by the interaction between RB and income uncertainty.[21] It is worth noting that this univariate RB model unique state variable s leads to the same consumption and saving functions as the corresponding multivariate RB model in which the state variables are w and y. The intuition behind this result is that the level of financial wealth w evolves deterministically over time, so that the evil agent cannot influence it.[22] Adopting the univariate setting here can significantly help solve the model explicitly when we consider state uncertainty into the RB model.

Expression (14) shows that the precautionary savings demand is increasing with the degree of robustness (ϑ) via increasing the value of $\widetilde{\gamma}$ and interacting with the fundamental uncertainty: labor income uncertainty (σ_s^2). An interesting question here is the relative importance of RB (ϑ) and CARA (γ) in determining the precautionary savings demand, holding other parameters constant. We can use the elasticities of precautionary saving as a measure of their importance. Specifically, using (14), we have the following proposition:

2 *The relative sensitivity of precautionary saving to robustness (RB, ϑ) and CARA (γ) can be measured by:*

$$\mu_{\gamma\vartheta} \equiv \frac{e_\gamma}{e_\vartheta} = \frac{1+\vartheta}{\vartheta} > 1, \tag{16}$$

where $e_\vartheta \equiv \frac{\partial\Gamma/\Gamma}{\partial\vartheta/\vartheta}$ *and* $e_\gamma \equiv \frac{\partial\Gamma/\Gamma}{\partial\gamma/\gamma}$ *are the elasticities of the precautionary saving demand to RB and CARA, respectively. (16) means that the precautionary savings demand is more sensitive to the coefficient of (absolute) risk aversion measured by γ than RB measured by ϑ.*

Proof. The proof is straightforward. ■

HST (1999) showed that the discount factor and the concern about robustness are observationally equivalent in the sense that they lead to the same consumption and investment decisions in

[21] Consequently, consumption is more sensitive to unanticipated shocks. See HST (1999) for a detailed discussion on how RB affects consumption and precautionary savings within the discrete-time LQG setting.

[22] The proof of the equivalence between the univaritae and multivariate RB models is available from the corresponding author by request.

a discrete-time LQG representative-agent permanent income model. The reason for this result is that introducing a concern about robustness increases savings in the same way as increasing the discount factor, so that the discount factor can be changed to offset the effect of a change in RB on consumption and investment.[23] In contrast, for our continuous-time CARA-Gaussian model discussed above, we have a more general observational equivalence result between δ, γ, and ϑ:

3 *Let*

$$\gamma^{fi} = \gamma (1 + \vartheta),$$

where γ^{fi} *is the coefficient of absolute risk aversion in the FI-RE model, consumption and savings are identical in the FI-RE and RB models, holding other parameter values constant. Furthermore, let* $\delta = r$ *in the RB model, and*

$$\delta^{fi} = r - \frac{1}{2}\vartheta (r\gamma)^2 \sigma_s^2,$$

where δ^{fi} *is the discount rate in the FI-RE model, consumption and savings are identical in the FI-RE and RB models, ceteris paribus.*

Proof. Using (12) and (14), the proof is straightforward. ■

2.4 Comparison with the Constraint Specification and the Multiple-Priors Utility Specification

Following HSTW (2006) and Hansen and Sargent (2007), we could use the following *constraint* specification of the above RB problem:

$$\sup_{c_t \in \mathcal{C}} \inf_{\upsilon_t} \left[-\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) + \upsilon(s_t) \sigma_s^2 J_s \right], \tag{17}$$

subject to

$$\frac{1}{2} (\upsilon(s_t) \sigma_s)^2 \leq \eta, \tag{18}$$

where $\eta > 0$ measures the consumer's tolerance for model misspecification. It is clear from the above constraint that the worst-case distortion is

$$\upsilon^*(s_t) = -\sqrt{2\eta}/\sigma_s < 0.$$

Substituting this expression into (11), we can easily solve for the consumption function. The following proposition summarizes the solution.

[23] As shown in HST (1999), the two models have different implications for asset prices because continuation valuations would alter as one alters the values of the discount factor and the robustness parameter within the observational equivalence set.

4 *Given η, the consumption function and the saving function are*

$$c_t^* = rs_t + \Psi - \Gamma, \tag{19}$$

and

$$d_t^* = f_t + \Gamma - \Psi, \tag{20}$$

respectively, where $f_t = \rho\left(y_t - \overline{y}\right) / \left(r + \rho\right)$ is the demand for savings "for a rainy day", $\Psi\left(r\right) = \left(\delta - r\right) / \left(r\gamma\right)$ captures the dissavings effect of relative impatience,

$$\Gamma \equiv \left(\frac{1}{2}r\gamma + \frac{\sqrt{2\eta}}{\sigma_s}\right)\sigma_s^2 \tag{21}$$

is the demand for precautionary savings due to the interaction of income uncertainty, risk aversion, and uncertainty aversion.

Proof. See Appendix 8.3. ■

Comparing (14) with (21), it is clear that the multiplier and constraint formulations are observationally equivalent in the sense that they lead to the same consumption and saving functions when the following restriction on ϑ and η is satisfied:

$$\vartheta = \frac{2\sqrt{2\eta}}{r\gamma\sigma_s} \text{ or } \eta = \frac{1}{8}\left(r\gamma\vartheta\sigma_s\right)^2. \tag{22}$$

If (22) holds, the two robustness formulations lead to different levels of the worst-case distortion: υ^* is $-2\sqrt{2\eta}/\sigma_s$ in the multiplier specification, whereas it is $-\sqrt{2\eta}/\sigma_s$ in the constraint specification. In contrast, if the same amount of distortion is perceived under the two robustness specification, i.e., $-r\gamma\vartheta = -\sqrt{2\eta}/\sigma_s$, the precautionary saving demand under the constraint specification is greater than that under the multiplier specification. From (22), we can also see that if we keep ϑ constant, η is "*elastic*" and will change accordingly when the stochastic properties of the income process change. For example, if the value of σ_s is reduced due to a stabilization taxation policy, the amount of model uncertainty (i.e., η) will be reduced.

As is well known, we can use either robust decision-making (Hansen and Sargent 2007) or recursive multiple-prior utility (Gilboa and Schmeidler 1989 and Chen and Epstein 2002) due to ambiguity aversion to capture the same idea that the decision maker is concerned about their model is misspecified and thus considers a range of models when making decisions. In a continuous-time setting, Chen and Epstein (2002) assume that under ambiguity, the agent's beliefs are captured by a set of probability measures equivalent to a reference probability measure. That is, the agent's belief can deviate from the reference probability measure within probability measures equivalent to it. We view the model proposed in Section 2.1 which has the subjective probability measure as the reference model. The reference model serves as a benchmark among all the candidate models that an ambiguity-averse agent is willing to consider. However, the agent doubts that the reference model is the true model governing the economy. He then considers a constrained set of alternative models that are sufficiently close to the reference model. Therefore, the basic idea of the multiple-priors utility specification is the same as that of the Hansen-Sargent robust decision making specification. It is straightforward to show that in our univariate setting, the robust control specifications (the multiplier and constraint specifications) and the multiple-priors utility specification lead to the same consumption and saving functions.[24] We can therefore conclude that the two different modeling devices are observationally equivalent in the sense that they lead to the same consumption-saving decisions as well as the general equilibrium outcome. For simplicity, in the subsequent analysis, we focus on the robust control specification.

3 General Equilibrium Implications of RB

3.1 Definition of the General Equilibrium

As in Huggett (1993) and Wang (2003), we assume that the economy is populated by a continuum of *ex ante* identical, but *ex post* heterogeneous agents, with each agent having the saving function, (14). In addition, we also assume that the risk-free asset in our model economy is a pure-consumption loan and is in zero net supply. It is worth noting that the key insights delivered in this paper may be also obtained in a CARA-Gaussian production economy with incomplete markets considered in Angeletos and Calvet (2006) by introducing a neoclassical production function and using capital and bond as the saving instruments. We consider the simpler Huggett-type endowment economy for two reasons. First, in the endowment economy, we can directly compare the model's predictions on the dynamics of individual consumption and income with its empirical counterpart, and do not need to infer the idiosyncratic productivity shock process. Second, the endowment economy allows us to solve the models explicitly, and thus helps us identify distinct channels via which RB interacts with risk aversion, discounting, and intertemporal substitution and affects the consumption-saving behavior.

In the model economy, the initial cross-sectional distribution of income is assumed to be its stationary distribution $\Phi(\cdot)$. By the law of large numbers in Sun (2006), provided that the spaces of agents and the probability space are constructed appropriately, aggregate income and the cross-sectional distribution of permanent income $\Phi(\cdot)$ are constant over time.

5 *The total savings demand "for a rainy day" in the precautionary savings model with RB equals zero for any positive interest rate. That is,* $F_t(r) = \int_{y_t} f_t(r)\, d\Phi(y_t) = 0$*, for* $r > 0$*.*

Proof. Given that labor income is a stationary process, the LLN can be directly applied and the proof is the same as that in Wang (2003). ■

[24] The proof is available from the corresponding author by request.

This proposition states that the total savings "for a rainy day" is zero, at any positive interest rate. Therefore, from (13), for $r > 0$, the expression for total savings under RB in the economy at time t can be written as:

$$D(\vartheta, r) \equiv \Gamma(\vartheta, r) - \Psi(r). \tag{23}$$

We can now define the equilibrium in our model as follows:

6 *Given (23), a general equilibrium under RB is defined by an interest rate r^* satisfying:*

$$D(\vartheta, r^*) = 0. \tag{24}$$

3.2 Theoretical Results

The following proposition shows the existence of the equilibrium and the PIH holds in the RB general equilibrium:

7 *There exists at least one equilibrium interest rate $r^* \in (0, \delta)$ in the precautionary-savings model with RB; if $\delta < \rho$ the equilibrium interest rate is unique on $(0, \delta)$. In equilibrium, each consumer's optimal consumption is described by the PIH, in that*

$$c_t^* = r^* s_t. \tag{25}$$

Furthermore, the evolution equations of wealth and consumption are

$$dw_t^* = f_t dt, \tag{26}$$

$$dc_t^* = \frac{r^*}{r^* + \rho} \sigma_y dB_t, \tag{27}$$

respectively, where $f_t = \rho(y_t - \overline{y}) / (r^* + \rho)$. Finally, the relative volatility of consumption growth to income growth is

$$\mu \equiv \frac{\mathrm{sd}(dc_t^*)}{\mathrm{sd}(dy_t)} = \frac{r^*}{r^* + \rho}. \tag{28}$$

Proof. If $r > \delta$, both $\Gamma(\vartheta, r)$ and $\Psi(r)$ in the expression for total savings $D(\vartheta, r^*)$ are positive, which contradicts the equilibrium condition: $D(\vartheta, r^*) = 0$. Since $\Gamma(\vartheta, r) - \Psi(r) < 0$ (> 0) when $r = 0$ $(r = \delta)$, the continuity of the expression for total savings implies that there exists at least one interest rate $r^* \in (0, \delta)$ such that $D(\vartheta, r^*) = 0$. To prove this equilibrium is unique, note that

$$\frac{\partial D(\vartheta, r)}{\partial r} = (1 + \vartheta)\gamma \frac{\sigma^2}{(r + \rho)^2}\left(\frac{1}{2} - \frac{r}{r + \rho}\right) + \frac{\delta}{r^2 \gamma}.$$

Let $r > 0$; the derivative is positive if

$$\rho > r.$$

Therefore, if $\rho > \delta$ there is only one equilibrium in $(0, \delta)$. From Expression (12), we can obtain the individual's optimal consumption rule under RB in general equilibrium as $c_t^* = r^* s_t$. Therefore, there exists a unique equilibrium in this aggregate economy. Substituting (25) into (3) yields (26). Using (4) and (25), we can obtain (27). ■

The intuition behind this proposition is similar to that in Wang (2003). With an individual's constant total precautionary savings demand $\Gamma(\vartheta, r)$, for any $r > 0$, the equilibrium interest rate r^* must be at a level with the property that individual's dissavings demand due to impatience is exactly balanced by their total precautionary-savings demand, $\Gamma(\vartheta, r^*) = \Psi(r^*)$. Following Caballero (1991) and Wang (2003, 2009), we set that $\gamma = 1.5$, $\sigma_y = 0.1823$, and $\rho = 0.0834$.[25] Figure 1 shows that the aggregate saving function $D(\vartheta, r)$ is increasing with the interest rate for different values of ϑ when $\delta = 0.03$, and there exists a unique interest rate r^* for every given ϑ such that $D(\vartheta, r^*) = 0$.[26]

Given (12), (14), and (24), it is clear that even though precautionary saving at the individual level increases with the degree of concerns about model misspecifications, the level of aggregate savings is equal to zero in the general equilibrium. That is, RB does not affect the level of aggregate wealth in the economy. Figure 1 shows how RB (ϑ) affects the equilibrium interest rate (r^*). It is clear from the figure that the stronger the preference for robustness, the less the equilibrium interest rate. From (28), we can see that RB can affect the volatility of consumption by reducing the equilibrium interest rate. The following proposition summarizes the results about how the persistence coefficient of income affects the impact of RB on the relative volatility:

8 *Using (28), we have:*

$$\frac{\partial \mu}{\partial \vartheta} = \frac{\rho}{(r^* + \rho)^2} \frac{\partial r^*}{\partial \vartheta} < 0$$

because $\rho > 0$ *and* $\partial r^* / \partial \vartheta < 0$.

Proof. The proof is straightforward. ■

In the next section, we will fully explore how RB affects the equilibrium interest rate and the equilibrium dynamics of consumption after estimating the income process and calibrating the RB parameter ϑ.

4 Quantitative Analysis

In this section, we first describe how we estimate the income process and calibrate the robustness parameter. We then present quantitative results on how RB affects the equilibrium interest rate

[25] In Section 4.1, we will provide more details about how to estimate the income process using the U.S. panel data. The main result here is robust to the choices of these parameter values.

[26] We ignore negative interest rate equilibria because the resulting consumption function does not make economic sense. It is easy to see that D has the same zeroes as a cubic function, so that there exist conditions under which the equilibrium is globally unique, but these conditions are not amenable to analysis.

and relative volatility of consumption to income.

4.1 Estimation of the Income Process

To implement the quantitative analysis, we need to first estimate the income process. That is, we need to estimate ρ and σ_y in the income process specification (1). We use micro data from the Panel Study of Income Dynamics (PSID). Following Blundell, Pistaferri, and Preston (2008), we define the household income as total household income (including wage, financial, and transfer income of head, wife, and all others in household) minus financial income (defined as the sum of annual dividend income, interest income, rental income, trust fund income, and income from royalties for the head of the household only) minus the tax liability of non-financial income. This tax liability is defined as the total tax liability multiplied by the non-financial share of total income. Tax liabilities after 1992 are not reported in the PSID and so we estimate them using the TAXSIM program from the NBER. Details on sample selection are reported in Appendix 8.1.

To exclude extreme outliers, following Floden and Lindé (2001) we then normalize both income and consumption measures as ratios of the mean of each year, and exclude household in bottom and top 1 percent of the distribution of those ratios. To eliminate possible heteroskedasticity in the income measures, we regress each on a series of demographic variables to remove variation caused by differences in age and education. We next subtract these fitted values from each measure to create a panel of income residuals. We then use this panel to estimate the household income process as specified by an stationary AR(1) process by running panel regressions on lagged income. Specifically, we specify the AR(1) process with Gaussian innovations as follows:

$$y_t = \phi_0 + \phi_1 y_{t-1} + \sigma\varepsilon_t,\ t \geq 1,\ \ |\phi_1| < 1, \tag{29}$$

where $\varepsilon_t \sim N(0,1)$, $\phi_0 = (1-\phi_1)\,\overline{y}$, $\overline{y}$ is the mean of y_t, and the initial level of labor income y_0 are given. Once we have estimates of ϕ_1 and σ, we can recover the drift and diffusion coefficients in the Ornstein-Uhlenbeck process specified in (1). This can be done by rewriting (29) in the time interval of $[t, t+\Delta t]$:[27]

$$y_{t+\Delta t} = \phi_0 + \phi_1 y_t + \sigma\sqrt{\Delta t}\varepsilon_{t+\Delta t}, \tag{30}$$

where $\phi_0 = \mu\left(1-\exp\left(-\rho\Delta t\right)\right)/\left(\rho\Delta t\right)$, $\phi_1 = \exp\left(-\rho\Delta t\right)$, $\sigma = \sigma_y\sqrt{\left(1-\exp\left(-2\rho\Delta t\right)\right)/\left(2\rho\Delta t\right)}$, and $\varepsilon_{t+\Delta t}$ is the time-$(t+\Delta t)$ standard normal distributed innovation to income. When the time interval, Δt, converges to 0, (30) reduces to the Ornstein-Uhlenbeck process, (1). The estimation results and the recovered persistence and volatility coefficients in (1) are reported in Table 2.

4.2 Calibration of the Robustness Parameter

To fully explore how RB affects the dynamics of consumption and labor income, we adopt the calibration procedure outlined in HSW (2002) and AHS (2003) to calibrate the value of the RB

[27] Note that here we use the fact that $\Delta B_t = \varepsilon_t\sqrt{\Delta t}$, where ΔB_t represents the increment of a Wiener process.

parameter (ϑ) that governs the degree of robustness. Specifically, we calibrate ϑ by using the method of detection error probabilities (DEP) that is based on a statistical theory of model selection. We can then infer what values of ϑ imply reasonable fears of model misspecification for empirically-plausible approximating models. The model detection error probability denoted by p is a measure of how far the distorted model can deviate from the approximating model without being discarded; low values for this probability mean that agents are unwilling to discard many models, implying that the cloud of models surrounding the approximating model is large. In this case, it is easier for the consumer to distinguish the two models. The value of p is determined by the following procedure. Let model P denote the approximating model, (4) and model Q be the distorted model, (8). Define p_P as

$$p_P = \text{Prob}\left(\ln\left(\frac{L_Q}{L_P}\right) > 0 \middle| P\right), \tag{31}$$

where $\ln\left(\frac{L_Q}{L_P}\right)$ is the log-likelihood ratio. When model P generates the data, p_P measures the probability that a likelihood ratio test selects model Q. In this case, we call p_P the probability of the model detection error. Similarly, when model Q generates the data, we can define p_Q as

$$p_Q = \text{Prob}\left(\ln\left(\frac{L_P}{L_Q}\right) > 0 \middle| Q\right). \tag{32}$$

Given initial priors of 0.5 on each model and the length of the sample is N, the detection error probability, p, can be written as:

$$p(\vartheta; N) = \frac{1}{2}(p_P + p_Q), \tag{33}$$

where ϑ is the robustness parameter used to generate model Q. Given this definition, we can see that $1 - p$ measures the probability that econometricians can distinguish the approximating model from the distorted model.

The general idea of the calibration procedure is to find a value of ϑ such that $p(\vartheta; N)$ equals a given value (for example, 20%) after simulating model P, (4), and model Q, (8).[28] In the continuous-time model with the iid Gaussian specification, $p(\vartheta; N)$ can be easily computed. Since both models P and Q are arithmetic Brownian motions with constant drift and diffusion coefficients, the log-likelihood ratios are Brownian motions and are normally distributed random variables. Specifically, the logarithm of the Radon-Nikodym derivative of the distorted model (Q) with respect to the approximating model (P) can be written as

$$\ln\left(\frac{L_Q}{L_P}\right) = \int_0^t \overline{\upsilon} dB_s - \frac{1}{2}\int_0^t \overline{\upsilon}^2 ds, \tag{34}$$

where

$$\overline{\upsilon} \equiv \upsilon^* \sigma_s = -r^* \vartheta \gamma \sigma_s. \tag{35}$$

[28] The number of periods used in the calculation, N, is set to be 31, the actual length of the data $(1980 - 2010)$.

Similarly, the logarithm of the Radon-Nikodym derivative of the approximating model (P) with respect to the distorted model (Q) is

$$\ln\left(\frac{L_P}{L_Q}\right) = -\int_0^t \overline{\upsilon} dB_s + \frac{1}{2}\int_0^t \overline{\upsilon}^2 ds. \tag{36}$$

Using (31)-(36), it is straightforward to derive $p(\vartheta; N)$:

$$p(\vartheta; N) = \Pr\left(x < \frac{\overline{\upsilon}}{2}\sqrt{N}\right), \tag{37}$$

where x follows a standard normal distribution. From the expressions of $\overline{\upsilon}$, (35), and $p(\vartheta; N)$, (37), it is clear that the value of p is decreasing with the value of ϑ. It is worth noting that under the observational equivalence condition between the multiplier and constraint robustness formulations, (37) can be rewritten as: $p(\vartheta; N) = \Pr\left(x < -\sqrt{2\eta}\sqrt{N}\right)$, where η is the upper bound on the distance between the two models and measures the consumer's tolerance for model misspecification.

We first explore the relationship between the DEP (p) and the value of the RB parameter, ϑ. A general finding is a negative relationship between these two variables. The left panel of Figure 2 illustrates how DEP (p) varies with the value of ϑ for different values of CARA (γ).[29] We can see from the figure that the stronger the preference for robustness (higher ϑ), the less the DEP (p) is. For example, let $\gamma = 2$, then $p = 0.412$ and $r^* = 3.05$ percent when $\vartheta = 1$, while $p = 0.156$ and $r^* = 2.11$ percent when $\vartheta = 6$.[30] Both values of p are reasonable as argued in AHS (2002), HSW (2002), Maenhout (2004), and Hansen and Sargent (Chapter 9, 2007). In other words, a value of ϑ between 1 and 6 is reasonable. Using (16), we have $\mu_{\gamma\vartheta} = 1.17$ and 2 when we set $p = 0.156$ and 0.412, respectively. That is, risk aversion is relatively more important than RB in determining the precautionary savings demand given plausibly calibrated values of ϑ, and the relative importance of risk aversion to RB decreases with the amount of model uncertainty.

The right panel of Figure 2 illustrates how DEP (p) varies with ϑ for different values of σ_y when γ equals 2.[31] It also shows that the higher the value of ϑ, the less the DEP (p). In addition, to calibrate the same value of p, less values of σ_y (i.e., more volatile labor income processes) leads to higher values of ϑ.[32] The intuition behind this result is that σ_s and ϑ have opposite effects on $\overline{\upsilon}$.

[29] Based on the estimation results, we set $\overline{y} = 1$, $\sigma_y = 0.182$, and $\rho = 0.083$. It is worth noting that the implied coefficient of relative risk aversion (CRRA) in our CARA utility specification can be written as: γc or γy. Given that the value of the CRRA is very stable and $\overline{\upsilon}$ can be expressed as $r\vartheta\gamma\sigma_y/(r+\rho)$, proportional changes in the mean and standard deviation of y do not change our calibration results because their impacts on γ and σ_y are just cancelled out. For example, if both $\overline{y}$ and σ_y are doubled, γ is reduced to half such that the product of γ and σ_y remains unchanged.

[30] Caballero (1990) and Wang (2009) also consider the $\gamma = 2$ case.

[31] Since $\sigma_s = \sigma_y/(r+\rho)$, both changes in the persistence coefficient (ρ) and changes in volatility coefficient (σ_y) will change the value of σ_s.

[32] It is straightforward to show that a reduction in ρ has similar impacts on the calibrated value of ϑ as an increase in σ_y.

(It is clear from (35).) To keep the same value of p, if one parameter increases, the other one must reduce to offset its effect on $\overline{v}$.

An important comment follows these calibration results. As emphasized in Hansen and Sargent (2007), in the robustness model, p can be used to measure the amount of model uncertainty, whereas ϑ is used to measure the degree of the agent's preference for RB. If we keep p constant when recalibrating ϑ for different values of γ, ρ, or σ_y, the amount of model uncertainty is held constant, i.e., the set of distorted models with which we surround the approximating model does not change. In contrast, if we keep ϑ constant, p will change accordingly when the values of γ, ρ, or σ_y change. That is, the amount of model uncertainty is "elastic" and will change accordingly when the fundamental factors change.

4.3 Effects of RB on the Equilibrium Interest Rate and Consumption Volatility

As shown in the theoretic results, the equilibrium interest rate and relative volatility of consumption to income are jointly determined by the degree of robustness, risk aversion, and the income process. To better see how RB affects the equilibrium interest rate and the relative volatility, we present two quantitative exercises here. The first exercise fixes the parameters of the income process at the estimated values and allows the risk aversion parameter to change, while the second exercise fixes risk aversion parameter and allows the key income process parameter to vary.

Figure 3 shows that the equilibrium interest rate and the equilibrium relative consumption volatility decrease with the calibrated value of ϑ for different values of γ when $\sigma_y = 0.182$, and $\rho = 0.083$. For example, when ϑ is increased from 1 to 4 (i.e., when p decreases from 0.432 to 0.275), r^* is reduced from 3.12 percent to 2.61 percent, and μ is reduced from 0.273 to 0.240 when $\gamma = 1.5$.[33] In addition, the figure also shows that the interest rate and the relative volatility decrease with γ for different values of ϑ.

Figure 4 shows that the equilibrium interest rate and the equilibrium relative consumption volatility decrease with the value of ϑ for different values of σ_y when $\gamma = 2$ and $\rho = 0.083$. The pattern of this figure is similar to that of Figure 3. In addition, the figure also shows that the interest rate and the relative volatility decrease with σ_y for different values of ϑ. For example, when σ_y increases from 0.182 to 0.23, r^* is reduced from 2.57 percent to 2.04 percent and μ is reduced from 0.234 to 0.197 when $\vartheta = 2$.

Our theoretical and quantitative results obtained above have the potential to explain the observed low real interest rate as well as the declines in the equilibrium real interest rate in the U.S. economy. One of our theoretical results shows that a larger concern about model uncertainty lowers the equilibrium real interest rate. In the U.S., the average real risk-free interest rate is about 1.87 percent between 1981 and 2010 if we use CPI to measure inflation, and is about 1.96 percent

[33]Note that in the FI-RE case, r^* is 3.37 percentage and μ is 0.289.

if we use PCE to measure inflation.[34] Therefore, depending on what inflation index is used, the risk-free rate is between 1.87 and 1.96 percent. In our following discussion, we set the risk free rate to be 1.91 percent which is the average of the two real interest rates under CPI and PCE. Using the equilibrium condition, it is straightforward to show that the full-information model without RB requires the coefficient of risk aversion parameter to be 6 to match this rate.[35] This value of CRRA seems too high for ordinary consumers. In most macroeconomic studies, the value of RRA is assumed to be from 1 to 3. In contrast, when consumers take into account model uncertainty, the model can generate an equilibrium interest rate of 1.91 percent with much lower values of the coefficient of risk aversion.[36] Figure 5 shows the relationship between γ and ϑ for the given real interest rate 1.91 percent and 1.56 percent, respectively.[37] For example, when $\gamma = 3$ and $\vartheta = 3$, the RB model leads to the same interest rate as in the FI model with $\gamma = 6$. Using the same calibration procedure discussed in Section 4.1, we find that the corresponding DEP is $p = 0.242$. In other words, agents have 24.2 percent probability that they cannot distinguish the distorted model from the approximating model. As argued in Hansen and Sargent (2007) and in Section 4.2, this value seems reasonable in the literature. In summary, incorporating model uncertainty due to RB can relax the restriction on CRRA imposed by the model and thus has the potential to explain the low interest rate we observed in the U.S. economy.

Specifically, the comparison between a model without model uncertainty (the FI-RE model) with a model taking into account model uncertainty (the RB model) shows agents' concern about model misspecification will increase aggregate savings and thus drives down the equilibrium interest rate. Furthermore, within the RB framework, we show an increase in the degree of model uncertainty will further reduce the equilibrium interest rate through increasing precautionary savings. The explanation that agents have become more concerned about model misspecification after the 2007–09 financial crisis is not unreasonable given the long and deep recession which generated skepticism about whether the standard macro models can fully capture how the economy is working. To provide a numerical example, under our calibrated parameter values and when $\gamma = 2$, an increase in model uncertainty reflected by a reduction in the DEP from $p = 0.345$ to $p = 0.138$ (or reduction in ϑ from 2 to 7) leads to a reduction in the equilibrium interest rate from 2.63 percent by about

[34]Following Campbell (2003), we calculate the average of the real 3-month Treasury yields. Here we choose the 1981 − 2010 period because it is more consistent with our sample period of the panel data in estimating the joint consumption and income process. When we consider an extended period from 1981 to 2015, the real interest rate is 1.37 percent when using CPI and is 1.75 percent when using PCE.

[35]Note that since we set the mean income level to be 1, the coefficient of relative risk aversion (CRRA) evaluated at this level is equal to the coefficient of absolute risk aversion (CARA).

[36]This result is comparable to that obtained in Barillas, Hansen, and Sargent (2009). They found that most of the observed high market price of uncertainty in the U.S. can be reinterpreted as a market price of model uncertainty rather than the traditional market price of risk.

[37]Here 1.56 percent real interest rate is the average of 1.37 percent (using CPI) and 1.75 percent (using PCE) for the period from 1981 to 2015.

1.91 percent. It is worth noting that the explanation of a lower equilibrium real interest rate due to higher savings is not new. Summers (2014) and Blanchard et al. (2014) also argue that increases in global savings could be a reason for a lower equilibrium real interest rate in the U.S. and other advanced economies. However, their explanations for higher savings usually rely on demographic trends (such as an aging population) and capital flows from emerging economies to advanced economies, while our explanation for increases in savings comes purely from agents' concern about model uncertainty. In addition, neither of these papers provides a structural model to quantify the effects, while we explicitly solve a stochastic general equilibrium model to show both the channel and the effect.

To examine how RB affects the relative volatility of consumption to income ($\mu = \mathrm{sd}\left(dc_t^*\right) / \mathrm{sd}\left(dy_t\right)$), we first followed Luo, Nie, Wang, and Young (2016) and constructed a panel data set which contains both consumption and income at the household level.[38] Figure 6 shows the relative volatility of consumption to income between 1980 and 2000.[39] From the figure, the average empirical value of the relative volatility (μ) is 0.377 for the $1980-1996$ period, and is 0.326 for the period from 1980 to 2010. The minimum and maximum values of the empirical relative volatility from 1980 to 2010 are 0.195 (year 2006) and 0.55 (year 1982), respectively. From the expression for the equilibrium relative volatility (28), we can see that when the real interest rate is low, it is difficult to generate sufficiently high relative volatility of consumption to income. For example, when $r^* = 1.91\%$, $\mu = 0.186$, which is below the empirical counterpart, 0.326, in the sample period. In Section 6, we will show that how recursive utility can help make the model explain this dimension better.

5 The Welfare Cost of Model Uncertainty

We can also quantify the effects of RB on the welfare cost of volatility in the general equilibrium using the Lucas elimination-of-risk method. (See Lucas 1987; Tallarini 2000).[40] It is worth noting that although we do not discuss the welfare costs of business cycles in our heterogeneous-agent economy without aggregate uncertainty, we can still use the Lucas approach to explore the welfare cost of model uncertainty due to RB.[41] Specifically, following the literature, we define the total

[38] Appendix 8.1 presents details on how the panel is constructed.

[39] Please see Appendix 8.1 for more details on how the panel was constructed.

[40] Tallarini (2000) found that the welfare costs of aggregate fluctuations are non-trivial when the representative agent has a recursive utility that breaks the link between risk aversion and intertemporal substitution. However, in Tallarini's model, high welfare costs also require the agent to have implausibly high levels of risk aversion. In contrast, Barillas, Hansen, and Sargent (2009) showed that the high coefficients of risk aversion in Tallarini (2000) may not only reflect the agent's risk attitudes but also reflect his concerns about model misspecification. They found that market prices of model uncertainty contain information about the benefits of removing model uncertainty, not the consumption fluctuations that Lucas (1987) studied.

[41] Ellison and Sargent (2014) found that idiosyncratic consumption risk has a greater impact on the cost of business cycles when they fear model misspecification. In addition, they showed that endowing agents with fears about misspecification leads to greater welfare costs caused by the exisiting idiosyncratic consumption risk.

welfare cost of volatility as the percentage of permanent income the consumer is ready to give up at the initial period to be as well off in the FI-RE economy as he is in the RB economy:[42]

$$\widetilde{J}(s_0(1-\Delta)) = J(s_0), \tag{38}$$

where

$$\widetilde{J}(s_0(1-\Delta)) = -\frac{1}{\widetilde{\alpha}_1}\exp(-\widetilde{\alpha}_0 - \widetilde{\alpha}_1 s_0(1-\Delta)) \text{ and } J(s_0) = -\frac{1}{\alpha_1}\exp(-\alpha_0 - \alpha_1 s_0)$$

are the value functions under FI-RE and RB, respectively, Δ is the compensating amount measured by the percentage of s_0, $\alpha_1 = r^*\gamma$, $\widetilde{\alpha}_1 = \widetilde{r}^*\gamma$, $\alpha_0 = \delta/r^* - 1 - (1+\vartheta)r^*\gamma^2\sigma_s^2/2$, $\widetilde{\alpha}_0 = \delta/\widetilde{r}^* - 1 - \widetilde{r}^*\gamma^2\widetilde{\sigma}_s^2/2$, and r^* and $\widetilde{r}^*$ are the equilibrium interest rates in the RB and FI-RE economies, respectively.[43] The following proposition summarizes the result about how RB affects the welfare costs in general equilibrium:

9 *When the equilibrium condition, (24), holds, the welfare costs due to model uncertainty can be written as:*

$$\Delta = \frac{s_0(\widetilde{\alpha}_1 - \alpha_1) - \ln(\widetilde{\alpha}_1/\alpha_1)}{\widetilde{\alpha}_1 s_0} = \left(1 - \frac{r^*}{\widetilde{r}^*}\right) - \frac{1}{\gamma\widetilde{c}_0}\ln\left(\frac{\widetilde{r}^*}{r^*}\right), \tag{39}$$

where $\widetilde{c}_0 = \widetilde{r}^* s_0$ is optimal consumption under FI-RE, which implies that

$$\frac{\partial\Delta}{\partial\vartheta} = \frac{\partial\Delta}{\partial r^*}\frac{\partial r^*}{\partial\vartheta} > 0$$

because $\partial r^*/\partial\vartheta < 0$, and $\partial\Delta/\partial r^* = -1/\widetilde{r}^*\left[1 - 1/(r^*\gamma s_0)\right] < 0$ for plausible parameter values.

Proof. Substituting (24) into the expressions of α_0 and $\widetilde{\alpha}_0$ in the value functions under FI-RE and RB, we obtain that $\alpha_0 = \widetilde{\alpha}_0 = 0$. Combining these results with (38) yields (39). ■

To do quantitative welfare analysis, we set s_0 such that $c_0 (= \widetilde{r}^* s_0) = y_0$. Figure 7 illustrates how the welfare cost of model uncertainty varies with ϑ for different values of γ and σ_y when $y_0 = 1$, $\gamma = 1.5$, and $\rho = 0.083$.[44] We can see from this figure that the welfare costs of model uncertainty are nontrivial and increasing in γ and σ_y. The intuition behind this result is that higher income uncertainty leads to higher induced model uncertainty. For example, when $\gamma = 2$ and $\vartheta = 2$, the welfare cost of model uncertainty Δ is 8.46%. If ϑ increases from 2 to 4, Δ increases from 8.46% percent to 12.23%. Furthermore, the figure also shows that an increase income volatility can significantly increase the welfare cost of model uncertainty. For example, when $\gamma = 2$, $\vartheta = 2$, and income volatility (σ_y) is increased from 0.18 to 0.23, Δ increases from 8.46% to 10.25%. One policy

[42] This approach is also used in Epaulard and Pommeret (2003) to examine the welfare cost of volatility in a representative-agent model with recursive utility. In their model, the total welfare cost of volatility is defined as the percentage of capital the representative agent is ready to give up at the initial period to be as well off in a certain economy as he is in a stochastic one.

[43] See Appendix 8.2 for the derivation of the value functions. Note that $\Delta = 0$ when $\vartheta = 0$.

[44] When generating the left and right panels of this figure, we set $\sigma_y = 0.182$ and $\gamma = 2$, respectively.

implication stemming from this finding is that macro policies aiming to reduce income volatility and inequality are more beneficial in an economy in which consumers have more fear about model uncertainty.

6 Stochastic Differential Utility under RB

In this section, we assume that consumers have stochastic differential utility (SDU, a continuous-time version of recursive utility) and thus risk aversion and intertemporal substitution are separated in their preferences. After solving the SDF model with RB explicitly, we discuss how the interaction of intertemporal substitution, risk aversion, and robustness affects individual consumption and savings decisions and the equilibrium interest rate.

6.1 SDU: Separation of Risk Aversion and Intertemporal Substitution

In the previous sections, we discussed how the interaction of risk aversion and robustness affects the equilibrium interest rate, consumption volatility, and welfare costs of model uncertainty. However, given the time-separable utility setting, we cannot examine how intertemporal substitution affects the equilibrium outcomes. In this section, following Duffie and Epstein (1992a), we consider a stochastic differential utility (SDU) model with constant intertemporal elasticity of substitution (CIES) and constant absolute risk aversion (CARA or exponential risk aversion).[45] This recursive utility (RU) specification can be viewed as a continuous-time version of the Weil (1993) model.

To obtain the consumption function and the value function under SDU, we start with a discrete-time setting and then consider the continuous-time limit of the discrete-time specification. Specifically, let Δt be a small discrete change in time. The diffusion process, (4), can be approximated as:

$$\Delta s_t \approx (rs_t - c_t)\,\Delta t + \sigma_s \Delta B_t, \tag{40}$$

where Δs_t is the change in s_t over the time interval Δt, $\Delta B_t = \sqrt{\Delta t}\epsilon$, and ϵ is a standard normal distributed variable. The corresponding Bellman equation for the optimization problem under RU can be written as:

$$J\left(s_t\right)^{1-1/\varepsilon} = \max_{c_t \in \mathcal{C}} \left\{ \left(1 - e^{-\delta \Delta t}\right) c_t^{1-1/\varepsilon} + e^{-\delta \Delta t} \mathcal{C}E_t^{1-1/\varepsilon} \right\} \tag{41}$$

subject to (4), where ε is the intertemporal elasticity of substitution, δ is the discount rate, γ is the coefficient of absolute risk aversion, and

$$\mathcal{C}E_t \equiv -\frac{1}{\gamma} \ln\left(E_t\left[\exp\left(-\gamma J\left(s_{t+\Delta t}\right)\right)\right]\right)$$

[45]SDU was introduced by Duffie and Epstein (2002a) as a continuous-time analog of recursive utility. For the applications of SDU with CIES and constant relative risk aversion (CRRA) in portfolio choice and asset pricing, See Svensson (1987), Duffie and Epstein (2002b), and Maenhout (2004).

denotes the certainty equivalent in terms of period-t consumption of the uncertain total utility in the future periods.[46] Furthermore, (41) can be reduced to

$$0=\max_{c_t}\left\{\delta c_t^{1-1/\varepsilon}-\delta\widetilde{J}(s_t)+\left(rs_t-c_t-\frac{1}{2}\gamma A\sigma_s^2\right)\widetilde{J}_s(s_t)\right\},$$

where $\widetilde{J}(s_t)=J(s_t)^{1-1/\varepsilon}=[A(s_t+A_0)]^{1-1/\varepsilon}$, and A and A_0 are undetermined coefficients.[47] (See Appendix 8.4 for the derivation.)

If the consumer trusts the model represented by (4), we can solve for the consumption function and the corresponding value function as follows:

$$c_t^*=[r+(\delta-r)\varepsilon](s_t+A_0)$$

and $J(s_t)=A(s_t+A_0)$, where $A=\left[\frac{r+(\delta-r)\varepsilon}{\delta^\varepsilon}\right]^{1/(1-\varepsilon)}$ and $A_0=-\gamma A\sigma_s^2/(2r)$.[48] Here we need to impose that $r+(\delta-r)\varepsilon>0$ to guarantee the existence of an optimal plan. In addition, as in Weil (1993), we also need to assume that the initial financial wealth level, w_0, is sufficiently high and the share of risky human wealth is sufficiently low in total wealth to guarantee that consumption would not become negative in finite time with positive probability.

6.2 Consumption and Saving Rules under RB

To introduce robustness into the above recursive utility model, we follow the same procedure as in the previous section and write the distorting model by adding an endogenous distortion $\upsilon(s_t)$ to the law of motion of the state variable s_t, (4),

$$ds_t=(rs_t-c_t)\,dt+\sigma_s(\sigma_s\upsilon(s_t)\,dt+dB_t)\,. \tag{42}$$

The drift adjustment $\upsilon(s_t)$ is chosen to minimize the sum of the expected continuation payoff, but adjusted to reflect the additional drift component in (42), and of an entropy penalty:

$$0=\sup_{c_t\in\mathcal{C}}\inf_{\upsilon_t}\left\{\delta c_t^{1-1/\varepsilon}-\delta\widetilde{J}(s_t)+\left(rs_t-c_t-\frac{1}{2}A\alpha\sigma_s^2\right)\widetilde{J}_s(s_t)+\sigma_s^2\upsilon_t\widetilde{J}_s(s_t)+\frac{1}{2\vartheta_t}\sigma_s^2\upsilon_t^2\right\},$$

where $\widetilde{J}(s_t)=[A(s_t+A_0)]^{1-1/\varepsilon}$ and $\widetilde{J}_s(s_t)=(1-1/\varepsilon)A[A(s_t+A_0)]^{-1/\varepsilon}$. The following proposition summarizes the solution to this RB problem:

[46] Kraft and Seifried (2014) provided a rigorous proof of the connection between the discrete-time RU specification and the SDU specification. Specifically, they showed that in a general semimartingale framework and under the standard assumption son the aggregator function, SDF is the continuous-time limit of RU.

[47] Note that here we use the fact that the log-exponenial operator can be simplified to:

$$\ln(E_t[\exp(-\gamma J(s_{t+dt}))])=-\gamma As_t-\gamma A_0-\gamma A(rs_t-c_t)\,dt+\frac{1}{2}\gamma^2A^2\sigma_s^2dt.$$

[48] Note that when $\delta=r$, i.e., the discount rate equals the interest rate, the intertemporal substittuion channel is shut down and the consumption rule reduces to: $c_t^*=rs_t-r\gamma\sigma_s^2/2$, which means that consumption is independent of intertemporal substitution in this special case.

10 *Given ϑ, the optimal consumption and saving functions under robustness are*

$$c_t^* = rs_t + \Psi_t - \Gamma, \tag{43}$$

$$d_t^* = f_t - \Psi_t + \Gamma, \tag{44}$$

respectively, where $f_t = \rho\,(y_t - \overline{y})\,/\,(r+\rho)$ is the demand for savings "for a rainy day",

$$\Psi_t \equiv (\delta - r)\,\varepsilon s_t \tag{45}$$

captures the dissavings effect of relative impatience,

$$\Gamma \equiv \frac{1}{2}\frac{A}{r}\left[r + (\delta - r)\,\varepsilon\right]\widetilde{\gamma}\sigma_s^2 \tag{46}$$

is the precautionary savings demand, $\widetilde{\gamma} \equiv \gamma + \vartheta$ is the effective coefficient of absolute risk aversion, and

$$A = \left[\frac{r + (\delta - r)\,\varepsilon}{\delta^{\varepsilon}}\right]^{1/(1-\varepsilon)}. \tag{47}$$

Proof. See Appendix 8.4. ■

When $\delta = r$, $A = r$ and this RU model is reduced to the benchmark model. The reason is that when the interest rate equals the discount rate, the effect of EIS on consumption growth and saving disappears. When $\delta \neq r$, A is increasing in ε. (We can see this from the upper panel of Figure 8.)[49] From (43), (45), and (46), we can see that EIS affects both the MPC out of s_t and the precautionary saving demand Γ when $\delta \neq r$. Specifically, both MPC and the precautionary saving demand increases with ε when $\delta > r$. The lower panel of Figure 8 clearly shows that Γ is increasing with ε. That is, the larger the elasticity of intertemporal substitution (i.e., the weaker the desire for consumption smoothing), the stronger the precautionary saving demand. The reason behind this result is clear from the following expression for expected consumption growth:

$$\frac{E_t\left[dc_t^*\right]}{dt} = \left[r + (\delta - r)\,\varepsilon\right]\left[\Gamma - (\delta - r)\,\varepsilon\right], \tag{48}$$

i.e., the precautionary saving demand leads to higher expected consumption growth. This result is consistent with that obtained in the discrete-time partial equilibrium RU model proposed by Weil (1993). It is worth noting that the OE between the discount factor and a concern about robustness established in HST (1999) also no longer holds in this RU model. It is clear from (43) to (46) that δ affect the MPC, $r + (\delta - r)\,\varepsilon$, whereas ϑ does not appear in the MPC.

[49] Empirical studies using aggregate data usually find the EIS to be close to zero, whereas calibrated RBC models usually require it to be close to one. For example, Hall (1988) found in the expected utility setting that the value of ε is close to 0.1. Guvenen (2006) allowed heterogeneity and estimated that the true value of ε is 0.47 in an economy with both stockholders who have high EIS and non-stockholders who have low EIS. Although theoretically we cannot rule out the $\varepsilon > 1$ case, we follow the literature and assume that $\varepsilon \leq 1$ in this paper.

The saving function, (44), can be decomposed as follows:

$$d_t^* = f_t - \Psi_{1,t} - \Psi_2 + \Gamma, \tag{49}$$

where

$$\Psi_{1,t} \equiv (\delta - r)\,\varepsilon\,(s_t - \overline{s}) \text{ and } \Psi_2 \equiv (\delta - r)\,\varepsilon\overline{s}.$$

The term, $\Psi_t = \Psi_{1,t} + \Psi_{2,t}$, captures the dissaving effect due to relative impatience, which is affine in the value of total source, the sum of financial wealth and human wealth. Furthermore, $\Psi_{1,t}$ is a mean reverting process and Ψ_2 is a constant term. It is worth noting that this part of saving measures consumers' intertemporal consumption smoothing motive, and is independent of the degree of risk aversion and labor income uncertainty. Unlike the benchmark model with the time-additive utility, in the RU case the Ψ_t term increases with the value of total wealth (s_t) when the consumers are relatively more impatient, i.e., $\delta > r$. This result is consistent with that obtained in Wang (2006) in which the dissaving effect is generated by the endogenous discount factor. In addition, the Ψ_t term can also capture the intuition that richer consumers are more impatient and thus dissave more in the long run used to model the endogenous discount factor.

6.3 General Equilibrium Implications

Using the individual saving function (49) and following the same aggregation procedure used in the previous sections, we have the following result on the total saving demand:

11 *Both the total demand of savings "for a rainy day" and the total demand for the estimation-risk-induced savings in the RB model with IC equal zero for any positive interest rate. That is,* $F_t(r) = \int_{y_t} f_t(r)\, d\Phi(y_t) = 0$ *and* $H_t(r) = \int_{s_t} \Psi_{1,t} d\Phi_s(s_t) = 0$*, for* $r > 0$.

Proof. The proof uses the LLN and is the same as that in Wang (2003). ■

This proposition states that the total savings "for a rainy day" is zero, at any positive interest rate. Therefore, from (49), after aggregating across all consumers, the expression for total savings in this RU model can be written as:

$$D(r) \equiv \Gamma(r) - \Psi_2(r), \tag{50}$$

where the first term measures the amount of precautionary savings due to risk aversion and uncertainty aversion, and the second term captures the steady state dissavings effects of impatience. As in the benchmark model, we define the equilibrium in our model as: $D(r^*) = 0$. The following proposition shows the existence of the equilibrium and the PIH holds in the general equilibrium:

12 *There exists at least one equilibrium with an interest rate* $r^* \in (0, \delta)$ *in the RB model with IC. In any such equilibrium, each consumer's optimal consumption is described by the PIH, in that*

$$c_t^* = [r^* + (\delta - r^*)\,\varepsilon]\, s_t - (\delta - r^*)\,\varepsilon\overline{s}. \tag{51}$$

Furthermore, in this equilibrium, the evolution equations of wealth and consumption are

$$dw_t^* = (f_t - \Psi_{1,t})\, dt, \tag{52}$$

$$dc_t^* = [r^* + (\delta - r^*)\,\varepsilon]\, ds_t, \tag{53}$$

respectively. Finally, the relative volatility of consumption growth to income growth is

$$\mu \equiv \frac{\mathrm{sd}\,(dc_t^*)}{\mathrm{sd}\,(dy_t)} = \frac{r^* + (\delta - r^*)\,\varepsilon}{r^* + \rho}. \tag{54}$$

Proof. If $r > \delta$, $D(\vartheta, r^*) > 0$ because $\Gamma > 0$ and $\Psi_2 < 0$, which contradicts the equilibrium condition: $D(\vartheta, r^*) = 0$. When $r = \delta$, it is straightforward to show that $\Gamma > 0$ and $\Psi_2 = 0$, which implies that Since $\Gamma - \Psi_2 > 0$. When r converges to 0, $\Psi_2 > 0$ and Γ converges to 0 because the value of A/r converges to 1, which implies that $\Gamma - \Psi_2 < 0$. The continuity of the expression for total savings thus implies that there exists at least one interest rate $r^* \in (0, \delta)$ such that $D(r^*) = \Gamma - \Psi_2 = 0$. ■

We can establish that uniqueness obtains on $(0, \delta)$ under a restriction that households are sufficiently close to expected utility.

13 *The equilibrium is unique if $\varepsilon > 0$ is small enough.*

Proof. We have

$$\frac{\partial D(\vartheta, r)}{\partial r} > 0$$

if

$$(2\varepsilon - 1)\, r^2 + (\rho - 3\delta\varepsilon)\, r - \delta\varepsilon\rho > 0.$$

There are no real roots of this quadratic if the discriminant is negative:

$$\Delta = (\rho - 3\delta\varepsilon)^2 + 4\,(2\varepsilon - 1)\,\delta\varepsilon\rho.$$

A necessary condition for $\Delta < 0$ is $0 < \varepsilon < 0.5$; thus, necessary conditions for uniqueness are

$$\varepsilon < 0.5,$$

$$(2\varepsilon - 1)\, r^2 + (\rho - 3\delta\varepsilon)\, r - \delta\varepsilon\rho > 0.$$

At $\varepsilon = 0$ the second condition reduces to

$$\rho > r,$$

which holds as before if $\rho > \delta$. By continuity these conditions continue to be satisfied for ε close enough to zero, so that D is monotonic on $(0, \delta)$. ■

Following the same calibration procedure adopted in Section 4.2, we can easily calibrate the value of ϑ using the DEP. Specifically, given that $\upsilon^* = -\vartheta A$, the DEP for this RU case, $p(\vartheta; N)$, can be expressed as:

$$p(\vartheta; N) = \Pr\left(x < \frac{\overline{\upsilon}}{2}\sqrt{N}\right), \tag{55}$$

where $\overline{v} \equiv v^* \sigma_s = -\vartheta A \sigma_s$. Since A increases with ε, (55) clearly shows that p decreases with ε for given values of ϑ. For example, when $\vartheta = 1.5$ and $\gamma = 1.5$, p decreases from $p = 0.438$ to 0.414 when ε increases from 0.1 to 0.4. That is, EIS does not have significant impacts on the amount of model uncertainty facing the consumer if we fix ϑ and allow for elastic model uncertainty. This result is not surprising because ε does not influence A significantly (we can see this from Figure 8).

Figure 9 shows that the aggregate saving function $D(r)$ is increasing with the interest rate, and there exists a unique interest rate r^* for different values of ε such that $D(r^*) = 0$.[50] From this figure, it is clear that that the equilibrium interest rate (r^*) increases with ε. That is, the larger the elasticity of intertemporal substitution, the larger the equilibrium interest rate. Comparing it with the result about the impact of EIS on the precautionary saving we obtained when r is given, it is clear that the intertemporal consumption smoothing motive measured by Ψ_2 dominates the precautionary saving motive in general equilibrium in the sense that an increase in ε has the potential to drive up the interest rate. Furthermore, the impact of ε on r^* is significant. For example, r^* increases from 1.27 percent to 1.88 percent as ε increases from 0.1 to 0.4.

Comparing (28) with (54), it is clear that the additional term, $(\delta - r^*)\,\varepsilon > 0$, due to the presence of RU has the potential to increase the relative volatility of consumption to income, and then make the model better explain the data in this crucial dimension and simultaneously keep the real interest rate at the low level. For example, when $\delta = 0.04$, $\varepsilon = 0.6$, $\gamma = 2$, $\sigma_s/\overline{s} = 5\%$, and $\vartheta = 5.6$, the corresponding DEP (p) is 0.269, the equilibrium interest rate (r^*) is 1.91 percent and the relative volatility (μ) is 0.31, which is just the empirical counterpart for the sample from 1980 to 2010.[51] When $\delta = 0.036$, $\varepsilon = 1.1$, $\gamma = 3$, $\sigma_s/\overline{s} = 5\%$, and $\vartheta = 6.5$, $p = 0.294$, $r^* = 1.91\%$, and $\mu = 0.37$, is just the empirical counterpart for the sample from 1980 to 1996. Note that these two values of p in the numerical examples are reasonable as argued in Hansen and Sargent (Chapter 9, 2007). In contrast, in the benchmark model, when $\delta = 0.04$ and $\gamma = 2$, we can still obtain that $r^* = 1.91\%$ when $\vartheta = 8$. But in this case, $\mu = 0.187$, which is well below the empirical counterpart.

7 Conclusions

This paper has developed a tractable continuous-time CARA-Gaussian framework to explore how model uncertainty due to robustness affects the interest rate and the dynamics of consumption and wealth in a general equilibrium heterogenous-agent economy. Using the explicit consumption-saving rules, we explored the relative importance of robustness and risk aversion in determining precautionary savings. Furthermore, we evaluated the quantitative effects of model uncertainty measured by the interaction of labor income uncertainty and calibrated values of the RB parameter on the general equilibrium interest rate, consumption volatility, and the welfare costs of model

[50] As in the benchmark mode, here we also set that $\gamma = 2$ and $\vartheta = 1.5$

[51] The conclusion is robust for different values of $\sigma_s/\overline{s}$.

uncertainty. Finally, we studied how RB interacts with stochastic differential utility and affect the equilibrium interest rate and consumption volatility.

8 Appendix

8.1 Description of Data

This appendix describes the data we use to estimate the income process as well as the method we use to construct a panel of both household income and consumption for our empirical analysis.

We use micro data from the Panel Study of Income Dynamics (PSID). Our household sample selection closely follows that of Blundell et al. (2008) as well.[52] We exclude households in the PSID low-income and Latino samples. We exclude household incomes in years of family composition change, divorce or remarriage, and female headship. We also exclude incomes in years where the head or wife is under 30 or over 65, or is missing education, region, or income responses. We also exclude household incomes where non-financial income is less than \$1000, where year-over-year income change is greater than \$90,000, and where year-over-year consumption change is greater than \$50,000. Our final panel contains 7,220 unique households with 54,901 yearly income responses and 50,422 imputed nondurable consumption values.[53]

The PSID does not include enough consumption expenditure data to create full picture of household nondurable consumption. Such detailed expenditures are found, though, in the Consumer Expenditure Survey (CEX) from the Bureau of Labor Statistics. But households in this study are only interviewed for four consecutive quarters and thus do not form a panel. To create a panel of consumption to match the PSID income measures, we use an estimated demand function for imputing nondurable consumption created by Guvenen and Smith (2014). Using an IV regression, they estimate a demand function for nondurable consumption that fits the detailed data in the CEX. The demand function uses demographic information and food consumption which can be found in both the CEX and PSID. Thus, we use this demand function of food consumption and demographic information (including age, family size, inflation measures, race, and education) to estimate nondurable consumption for PSID households, creating a consumption panel.

In order to estimate the income process, we narrow the sample period to the years $1980-1996$, due to the PSID survey changing to a biennial schedule after 1996. To further restrict the sample to exclude households with dramatic year-over-year income and consumption changes, we eliminate

[52] They create a new panel series of consumption that combines information from PSID and CEX, focusing on the period when some of the largest changes in income inequality occurred.

[53] There are more household incomes than imputed consumption values because food consumption - the main input variable in Guvenen and Smith's nondurable demand function - is not reported in the PSID for the years 1987 and 1988. Dividing the total income responses by unique households yields an average of $7-8$ years of responses per household. These years are not necessarily consecutive as our sample selection procedure allows households to be excluded in certain years but return to the sample if they later meet the criteria once again.

household observations in years where either income or consumption has increased more than 200 percent or decreased more than 80 percent from the previous year.

8.2 Solving the Benchmark RB Model

The Bellman equation associated with the optimization problem is

$$J(s_t) = \sup_{c_t} \left[-\frac{1}{\gamma} \exp(-\gamma c_t) + \exp(-\delta dt) J(s_{t+dt}) \right],$$

subject to (8), where $J(s_t)$ is the value function. The Hamilton-Jacobi-Bellman (HJB) equation for this problem is then

$$0 = \sup_{c_t} \left[-\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) \right],$$

where $\mathcal{D}J(s_t) = J_s(rs_t - c_t) + \frac{1}{2} J_{ss}\sigma_s^2$. Under RB, the HJB can be written as

$$\sup_{c_t} \inf_{\upsilon_t} \left[-\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + \mathcal{D}J(s_t) + \upsilon(s_t)\sigma_s^2 J_s + \frac{1}{2\vartheta(s_t)} \upsilon^2(s_t)\sigma_s^2 \right]$$

subject to the distorting equation, (8). Solving first for the infimization part of the problem yields

$$\upsilon^*(s_t) = -\vartheta(s_t) J_s.$$

Given that $\vartheta(s_t) > 0$, the perturbation adds a negative drift term to the state transition equation because $J_s > 0$. Substituting for υ^* in the robust HJB equation gives:

$$\sup_{c_t} \left[-\frac{1}{\gamma} \exp(-\gamma c_t) - \delta J(s_t) + (rs_t - c_t) J_s + \frac{1}{2}\sigma_s^2 J_{ss} - \frac{1}{2}\vartheta(s_t)\sigma_s^2 J_s^2 \right]. \tag{56}$$

Performing the indicated optimization yields the first-order condition for c_t:

$$c_t = -\frac{1}{\gamma} \ln(J_s). \tag{57}$$

Substituting (57) back into (56) to arrive at the partial differential equation (PDE):

$$0 = -\frac{J_s}{\gamma} - \delta J + \left(rs_t + \frac{1}{\gamma} \ln(J_s) \right) J_s + \frac{1}{2} \left(J_{ss} - \vartheta_t J_s^2 \right) \sigma_s^2. \tag{58}$$

Conjecture that the value function is of the form

$$J(s_t) = -\frac{1}{\alpha_1} \exp(-\alpha_0 - \alpha_1 s_t),$$

where α_0 and α_1 are constants to be determined. Using this conjecture, we obtain that $J_s = \exp(-\alpha_0 - \alpha_1 s_t) > 0$ and $J_{ss} = -\alpha_1 \exp(-\alpha_0 - \alpha_1 s_t) < 0$, and guess that

$$\vartheta(s_t) = -\frac{\vartheta}{J(s_t)} = \frac{\alpha_1 \vartheta}{\exp(-\alpha_0 - \alpha_1 s_t)} > 0.$$

(58) can thus be reduced to

$$-\delta\frac{1}{\alpha_1}=-\frac{1}{\gamma}+\left[rs_t-\left(\frac{\alpha_0}{\gamma}+\frac{\alpha_1}{\gamma}s_t\right)\right]-\frac{1}{2}\alpha_1\left(1+\vartheta\right)\sigma_s^2.$$

Collecting terms, the undetermined coefficients in the value function turn out to be

$$\alpha_1=r\gamma \text{ and } \alpha_0=\frac{\delta}{r}-1-\frac{1}{2}\left(1+\vartheta\right)r\gamma^2\sigma_s^2.$$

Substituting them back into the first-order condition (57) yields the consumption function, (12), in the main text.

Finally, we check if the consumer's transversality condition (TVC),

$$\lim_{t\to\infty}E\left[\exp\left(-\delta t\right)\left|J\left(s_t\right)\right|\right]=0, \tag{59}$$

is satisfied. Substituting the consumption function, c_t^*, into the state transition equation for s_t yields:

$$ds_t=Adt+\sigma dB_t,$$

where $A=-\frac{\delta-r}{r\gamma}+\frac{1}{2}r\widetilde{\gamma}\sigma_s^2$ under the approximating model. This Brownian motion with drift can be rewritten as:

$$s_t=s_0+At+\sigma\left(B_t-B_0\right), \tag{60}$$

where $B_t-B_0\sim N\left(0,t\right)$. Substituting (60) into $E\left[\exp\left(-\delta t\right)\left|J\left(s_t\right)\right|\right]$ yields:

$$\begin{aligned}E\left[\exp\left(-\delta t\right)\left|J\left(s_t\right)\right|\right]&=\frac{1}{\alpha_1}E\left[\exp\left(-\delta t-\alpha_0-\alpha_1 s_t\right)\right]\\&=\frac{1}{\alpha_1}\exp\left(E\left[-\delta-\alpha_0-\alpha_1 s_t\right]+\frac{1}{2}\operatorname{var}\left(\alpha_1 s_t\right)\right)\\&=\frac{1}{\alpha_1}\exp\left(-\delta t-\alpha_0-\alpha_1\left(s_0+At\right)+\frac{1}{2}\alpha_1^2\sigma^2 t\right)\\&=\left|J\left(s_0\right)\right|\exp\left(-\left(\delta+\alpha_1 A-\frac{1}{2}\alpha_1^2\sigma^2\right)t\right)\end{aligned}$$

where $\left|J\left(s_0\right)\right|=\frac{1}{\alpha_1}\exp\left(-\alpha_0-\alpha_1 s_0\right)$ is a positive constant and we use the facts that $s_t-s_0\sim N\left(At,\sigma^2 t\right)$. Therefore, the TVC, (59), is satisfied if and only if the following condition holds:

$$\delta+\alpha_1 A-\frac{1}{2}\alpha_1^2\sigma^2=r+\frac{1}{2}\left(r\gamma\right)^2\vartheta\sigma_s^2>0.$$

Given the parameter values we consider in the text, it is obvious that the TVC is always satisfied in both the FI-RE and RB models. It is straightforward to show that the TVC still holds under the distorted model in which $A=-\frac{\delta-r}{r\gamma}+\frac{1}{2}r\widetilde{\gamma}\sigma_s^2-r\gamma\vartheta\sigma_s^2$ for plausible values of ϑ.

8.3 Solving the Constraint Version of the RB Model

Substituting for $\upsilon^* = -\sqrt{2\eta}/\sigma_s$ in the robust HJB equation gives:

$$\sup_{c_t} \left[-\frac{1}{\gamma} \exp\left(-\gamma c_t\right) - \delta J\left(s_t\right) + \left(r s_t - c_t - \sqrt{2\eta}\sigma_s \right) J_s + \frac{1}{2}\sigma_s^2 J_{ss} \right]. \tag{61}$$

Performing the indicated optimization yields the first-order condition for c_t:

$$c_t = -\frac{1}{\gamma} \ln\left(J_s\right). \tag{62}$$

Substituting (62) back into (61) to arrive at the partial differential equation (PDE):

$$0 = -\frac{J_s}{\gamma} - \delta J + \left(r s_t + \frac{1}{\gamma} \ln\left(J_s\right) - \sqrt{2\eta}\sigma_s \right) J_s + \frac{1}{2}\sigma_s^2 J_{ss}. \tag{63}$$

Conjecture that the value function is of the form

$$J\left(s_t\right) = -\frac{1}{\alpha_1} \exp\left(-\alpha_0 - \alpha_1 s_t\right),$$

where α_0 and α_1 are constants to be determined. Using this conjecture, (63) can thus be reduced to

$$-\delta \frac{1}{\alpha_1} = -\frac{1}{\gamma} + \left[r s_t - \left(\frac{\alpha_0}{\gamma} + \frac{\alpha_1}{\gamma} s_t \right) - \sqrt{2\eta}\sigma_s \right] - \frac{1}{2}\alpha_1 \sigma_s^2.$$

Collecting terms, the undetermined coefficients in the value function turn out to be

$$\alpha_1 = r\gamma \text{ and } \alpha_0 = \frac{\delta}{r} - 1 - \left(\frac{1}{2} r\gamma + \frac{\sqrt{2\eta}}{\sigma_s} \right) \gamma \sigma_s^2.$$

Substituting them back into the first-order condition (62) yields the consumption function, (19), in the main text.

8.4 Solving the RB Model with Recursive Utility

We first guess that the value function is $J\left(s_t\right) = A s_t + A_0$. The value function at t time $t + \Delta t$ can thus be written as $J\left(s_{t+\Delta t}\right) = A s_{t+\Delta t} + A_0$ and the change in the value function is

$$\Delta J \equiv J\left(s_{t+\Delta t}\right) - J\left(s_t\right) = A \Delta s_t \approx A\left(r s_t - c_t\right)\Delta t + A\sigma_s \Delta B_t,$$

where $\Delta s_t \equiv s_{t+\Delta t} - s_t$. Furthermore,

$$\begin{aligned} E_t\left[\exp\left(-\gamma J\left(s_{t+\Delta t}\right)\right)\right] &= E_t\left[\exp\left(-\gamma A s_t - \gamma A_0 - \gamma A\left(r s_t - c_t\right)\Delta t - \gamma A \sigma_s \Delta B_t\right)\right] \\ &= \exp\left(-\gamma A s_t - \gamma A_0\right) \exp\left(-\gamma A\left(r s_t - c_t\right)\Delta t\right) \exp\left(\frac{1}{2}\gamma^2 A^2 \sigma_s^2 \Delta t\right), \end{aligned}$$

where we use the fact that $\Delta B_t = \sqrt{\Delta t}\epsilon$ and ϵ is a standard normal distributed variable. We can therefore obtain:

$$-\frac{1}{\gamma}\ln\left(E_t\left[\exp\left(-\gamma J\left(s_{t+\Delta t}\right)\right)\right]\right) = As_t + A_0 + A\left(rs_t - c_t\right)\Delta t - \frac{1}{2}\gamma A^2\sigma_s^2\Delta t.$$

Substituting this expression back into the Bellman equation yields:

$$J\left(s_t\right)^{1-1/\varepsilon} = \sup_{c_t}\left\{\left(1-e^{-\delta\Delta t}\right)c_t^{1-1/\varepsilon} + e^{-\delta\Delta t}\left[J\left(s_t\right) + A\left(rs_t - c_t\right)\Delta t - \frac{1}{2}\gamma A^2\sigma_s^2\Delta t\right]^{1-1/\varepsilon}\right\}.$$

Dividing both sides of this equation by $J\left(s_t\right)^{1-1/\varepsilon}$ and guessing that $c_t = \varphi J\left(s_t\right)$ yields

$$0 = \sup_{c_t}\left\{\delta\varphi^{1-1/\varepsilon}\Delta t + \left(1-\frac{1}{\varepsilon}\right)\frac{A\left(rs_t - c_t\right) - 0.5\gamma A^2\sigma_s^2}{J\left(s_t\right)}\Delta t - \delta\Delta t\right\}$$

Dividing it by yields and allowing $\Delta t \to 0$:

$$0 = \sup_{\varphi}\left\{\delta\varphi^{1-1/\varepsilon} + \left(1-\frac{1}{\varepsilon}\right)\left(-A\varphi + r\right) - \delta\right\}$$

The FOC is

$$\varphi = \left(\frac{A}{\delta}\right)^{-\varepsilon}.$$

Substituting it back into the HJB equation yields:

$$A = \delta\left[\frac{r + \varepsilon\left(\delta - r\right)}{\delta}\right]^{1/(1-\varepsilon)}$$

Under RB, the HJB can be written as:

$$0 = \sup_{c_t}\inf_{\upsilon_t}\left\{\delta c_t^{1-1/\varepsilon} - \delta\widetilde{J}\left(s_t\right) + \left(rs_t - c_t - \frac{1}{2}A\gamma\sigma_s^2\right)\widetilde{J}_s\left(s_t\right) + \sigma_s^2\upsilon_t\widetilde{J}_s\left(s_t\right) + \frac{1}{2\vartheta_t}\sigma_s^2\upsilon_t^2\right\},$$

where $\widetilde{J}\left(s_t\right) = \left[A\left(s_t + A_0\right)\right]^{1-1/\varepsilon}$ and $\widetilde{J}_s\left(s_t\right) = \left(1-1/\varepsilon\right)A\left[A\left(s_t + A_0\right)\right]^{-1/\varepsilon}$. In addition, we assume that $\vartheta_t = -\vartheta A/\widetilde{J}_s$ to guarantee the homothecity of the RB problem. Solving first for the infimization part of the problem yields

$$\upsilon_t^* = -\vartheta_t\widetilde{J}_s = \vartheta A.$$

Substituting υ_t^* back into the above robust HJB equation and following the same procedure above, we obtain:

$$A = \left[\frac{r + \left(\delta - r\right)\varepsilon}{\delta^{\varepsilon}}\right]^{1/(1-\varepsilon)} \text{ and } A_0 = -\frac{1}{2}\frac{A}{r}\left(\gamma + \vartheta\right)\sigma_s^2.$$

References

[1] Anderson, Evan W., Lars Peter Hansen, and Thomas J. Sargent (2003), "A Quartet of Semigroups for Model Specification, Robustness, Prices of Risk, and Model Detection," *Journal of the European Economic Association*, 1(1), 68-123.

[2] Angeletos, George-Marios and Laurent-Emmanuel Calvet (2006). "Idiosyncratic Production Risk, Growth and the Business Cycle," *Journal of Monetary Economics* 53, 1095-1115.

[3] Barillas, Francisco, Lars Peter Hansen, and Thomas J. Sargent (2009), "Doubts or Variability?," *Journal of Economic Theory* 144(6), 2388–2418.

[4] Bewley, Truman (1986), "Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers," in W. Hildenbrand and A. Mas-Colell, eds., *Contributions to mathematical economics: In honor of Gerard Debreu.* New York: North-Holland, 79–102.

[5] Bidder, Rhys and Matthew Smith (2012), "Robust Animal Spirits," *Journal of Monetary Economics* (59)8, 738–750.

[6] Blanchard, Olivier, Davide Furceri and Andrea Pescatori (2014), "A prolonged period of low real interest rates?," *Secular Stagnation: Facts, Causes and Cures*, Edited by Coen Tuulings and Richard Baldwin, CEPR Press, 2014.

[7] Blundell Richard, Luigi Pistaferri, and Ian Preston (2008), "Consumption Inequality and Partial Insurance," *American Economic Review* 98(5), 1887-1921.

[8] Caballero, Ricardo (1990), "Consumption Puzzles and Precautionary Savings," *Journal of Monetary Economics* 25(1), 113-136.

[9] Cagetti, Marco, Lars Peter Hansen, Thomas J. Sargent, and Noah Williams (2002), "Robustness and Pricing with Uncertain Growth," *Review of Financial Studies* 15, 363-404.

[10] Chen, Zengjing and Larry Epstein (2002), "Ambiguity, Risk, and Asset Returns in Continuous Time," *Econometrica* 70, 1403-1443.

[11] Duffie, Darrell and Epstein, Larry (1992), "Stochastic Differential Utility," *Econometrica* 60(2), 353-94.

[12] Duffie, Darrell and Epstein, Larry (1992), "Asset Pricing with Stochastic Differential Utility," *Review of Financial Studies* 5(3), 411-36.

[13] Ellison, Martin and Thomas J. Sargent (2014), "Welfare Cost of Business Cycles in Economies with Individual Consumption Risk," forthcoming in *American Economic Journal: Macroeconomics.*

[14] Epaularda, Anne and Aude Pommeret (2003), "Recursive utility, Growth, and the Welfare Cost of Volatility," *Review of Economic Dynamics* 6(3), 672-684.

[15] Floden, Martin and Jesper Lindé (2001), "Idiosyncratic Risk in the United States and Sweden: Is There a Role for Government Insurance?" *Review of Economic Dynamics* 4, 406-437

[16] Guvenen, Fatih (2006), "Reconciling Conflicting Evidence on the Elasticity of Intertemporal Substitution: A Macroeconomic Perspective," *Journal of Monetary Economics* 53(7), 1451-1472.

[17] Guvenen, Fatih and Anthony A. Smith, Jr., (2014), "Inferring Labor Income Risk and Partial Insurance from Economic Choices," forthcoming in *Econometrica*.

[18] Hall, Robert E. (1978), "Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence," *Journal of Political Economy* 86(6), 971-87.

[19] Hall, Robert E. (1988), "Intertemporal Substitution in Consumption," *Journal of Political Economy* 96(2), 339-57.

[20] Hamilton, James D., Ethan S. Harris, Jan Hatzius, and Kenneth D. West (2015), "The Equilibrium Real Funds Rate: Past, Present and Future," working paper.

[21] Hansen, Lars Peter and Thomas J. Sargent (1995), "Discounted Linear Exponential Quadratic Gaussian Control," *IEEE Transactions on Automatic Control* 40, 968-71.

[22] Hansen, Lars Peter and Thomas J. Sargent (2007), *Robustness*, Princeton University Press.

[23] Hansen, Lars Peter, Thomas J. Sargent, and Thomas D. Tallarini, Jr. (1999), "Robust Permanent Income and Pricing," *Review of Economic Studies* 66(4), 873-907.

[24] Hansen, Lars Peter, Thomas J. Sargent, and Neng Wang (2002), "Robust Permanent Income and Pricing with Filtering," *Macroeconomic Dynamics* 6(1), 40-84.

[25] Hansen, Lars Peter, Thomas J. Sargent, Gauhar Turmuhambetova, and Noah Williams (2006), "Robust Control and Model Misspecification," *Journal of Economic Theory* 128(1), 45-90.

[26] Huggett, Mark (1993), "The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control* 17(5-6), 953-969.

[27] Ilut, Cosmin and Scheinder, Martin (2014), "Ambiguous Business Cycles," *American Economic Review* 104(8), 2368-99.

[28] Kasa, Kenneth (2006), "Robustness and Information Processing," *Review of Economic Dynamics* 9(1), 1-33.

[29] Kraft, Holger and Frank Seifried (2014), "Stochastic Differential Utility as the Continuous-Time Limit of Recursive Utility," *Journal of Economic Theory* 151, 528-550.

[30] Lei, Chon Io (2001), "Why don't Investors Have Large Positions in Stocks?: A Robustness Perspective," Manuscript, University of Chicago.

[31] Liu, Jun, Jun Pan, and Tang Wang (2005), "An Equilibrium Model of Rare-Event Premia and Its Implication for Option Smirks," *Review of Financial Studies* 18(1), 131-164.

[32] Lucas, Robert E. (1987), *Models of Business Cycles*, New York: Basil Blackwell.

[33] Luo, Yulei (2008), "Consumption Dynamics under Information Processing Constraints" *Review of Economic Dynamics* 11(2), 366-385.

[34] Luo, Yulei and Eric R. Young (2010), "Risk-sensitive Consumption and Savings under Rational Inattention," *American Economic Journal: Macroeconomics* 2(4), 281-325.

[35] Luo, Yulei, Jun Nie, and Eric R. Young (2012), "Robustness, Information-Processing Constraints, and the Current Account in Small Open Economies," *Journal of International Economics* 88(1), 104-120.

[36] Maenhout, Pascal (2004), "Robust Portfolio Rules and Asset Pricing," *Review of Financial Studies* 17, 951–983.

[37] Summers, Larry (2014), "U.S. Economic Prospects: Secular Stagnation, Hysteresis, and the Zero Lower Bound," *Business Economics* 49(2).

[38] Svensson, Lars (1989), "Portfolio Choice with Non-expected Utility in Continuous Time," *Economics Letters* 30(4), 313-317.

[39] Tallarini, Thomas D. (2000), "Risk-sensitive Real Business Cycles," *Journal of Monetary Economics* 45(3), 507-532.

[40] Wang, Neng (2003), "Caballero Meets Bewley: the Permanent-Income Hypothesis in General Equilibrium," *American Economic Review* 93(3), 927-936.

[41] Wang, Neng (2006), "An Equilibrium Model of Wealth Distribution," *Journal of Monetary Economics* 54(7), 1882-1904.

[42] Young, Eric R. (2012), "Robust Policymaking in the Face of Sudden Stops," *Journal of Monetary Economics* 59(5), 512-527.

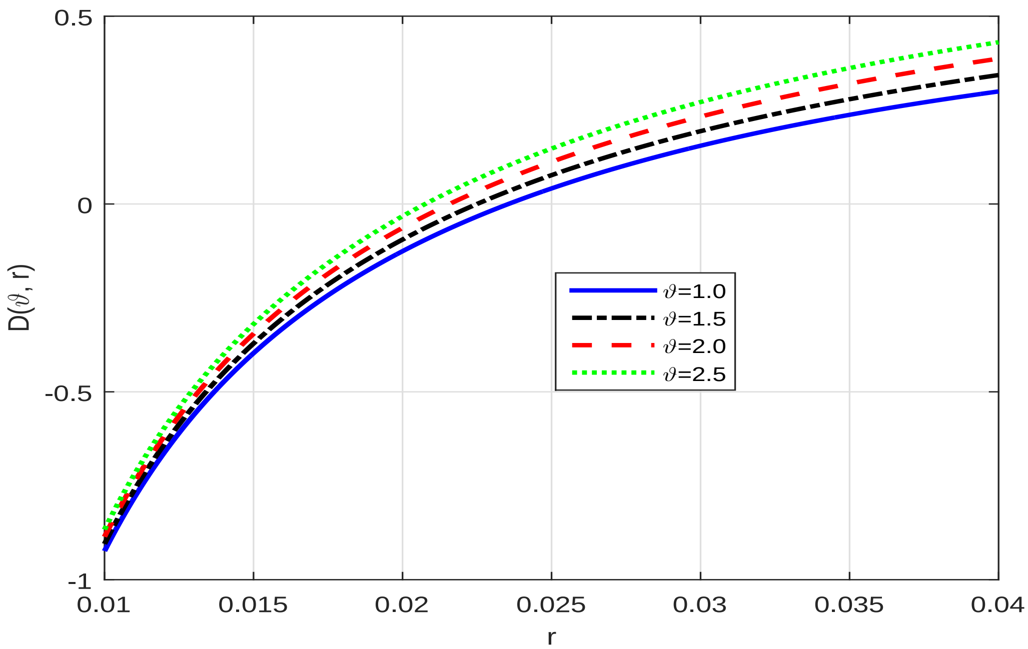

Figure 1: Effects of RB on Aggregate Savings

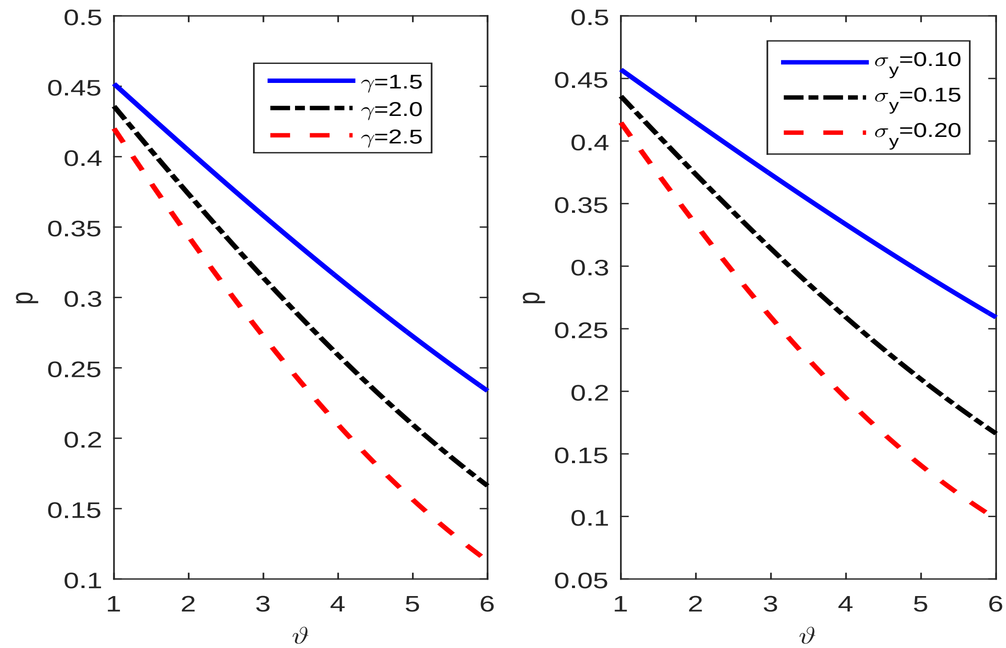

Figure 2: Relationship between ϑ and p

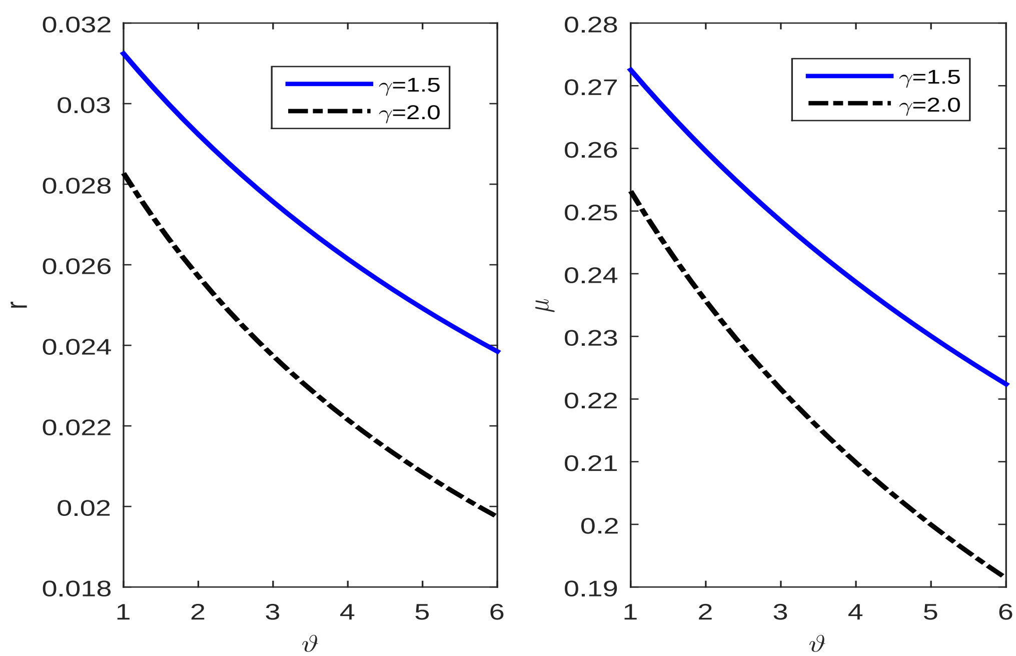

Figure 3: Effects of RB on the Interest Rate and Consumption Volatility

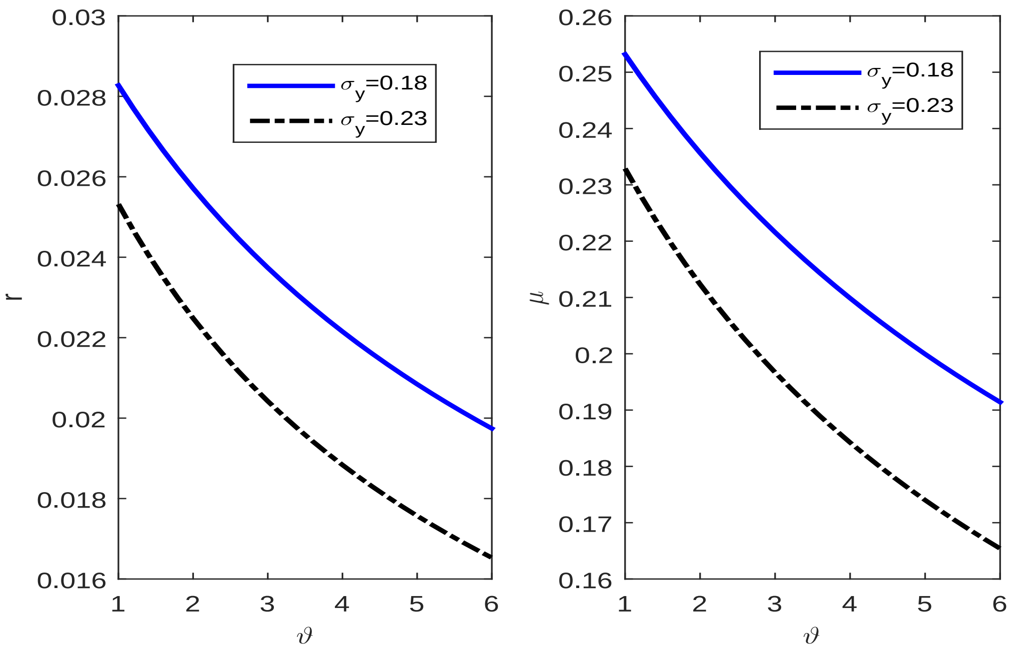

Figure 4: Effects of RB on the Interest Rate and Consumption Volatility

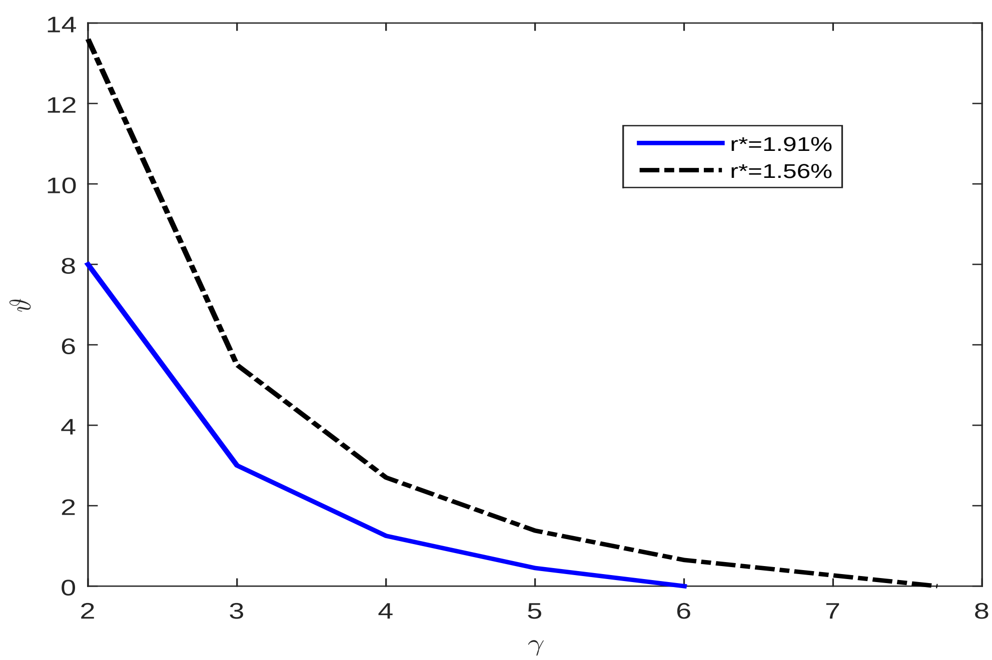

Figure 5: Relationship between γ and ϑ.

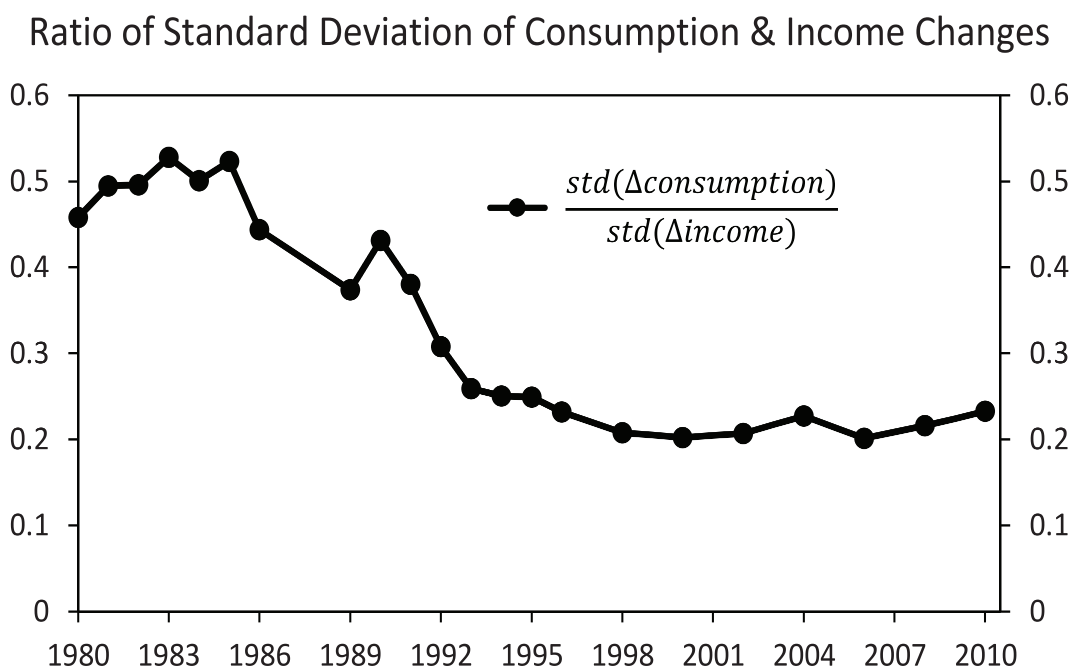

Figure 6: Relative Consumption Dispersion

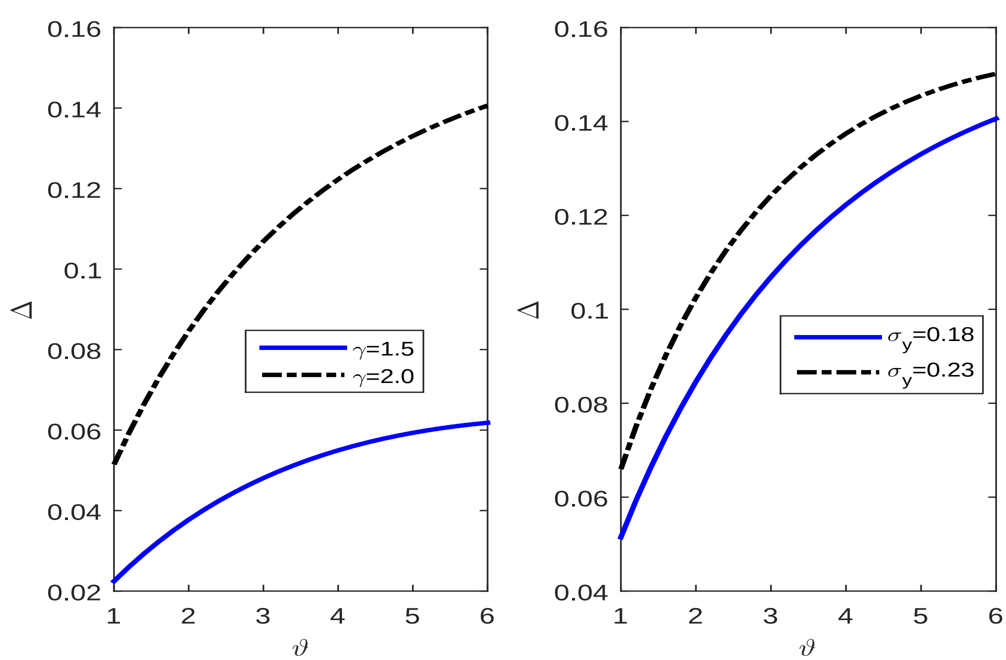

Figure 7: Effects of RB on the Welfare Cost

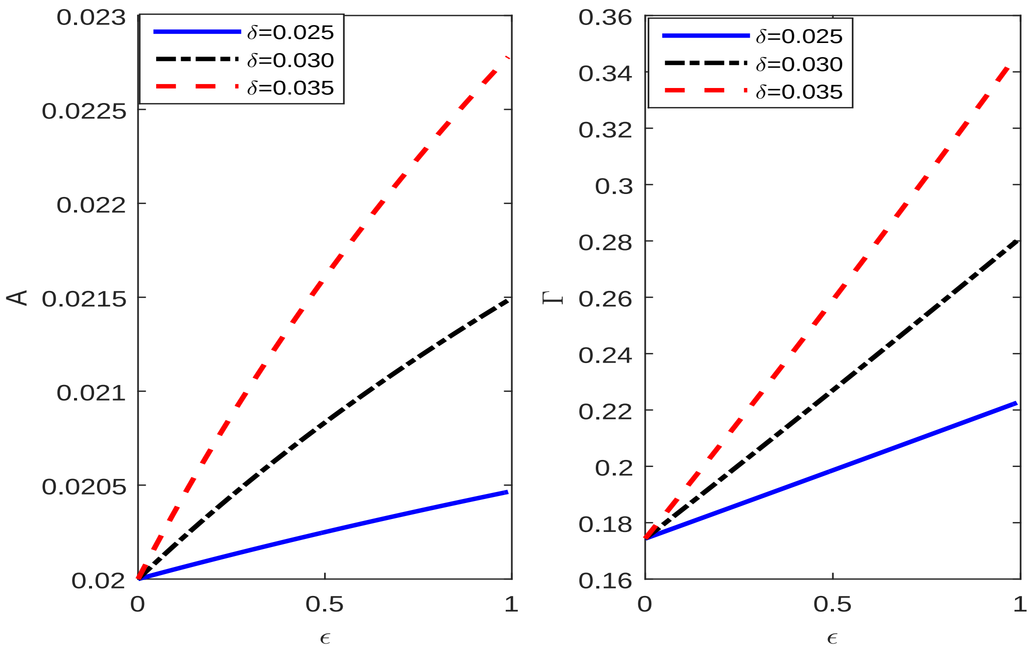

Figure 8: Effects of EIS on A and Γ in Partial Equilibrium (when $r = 2\%$)

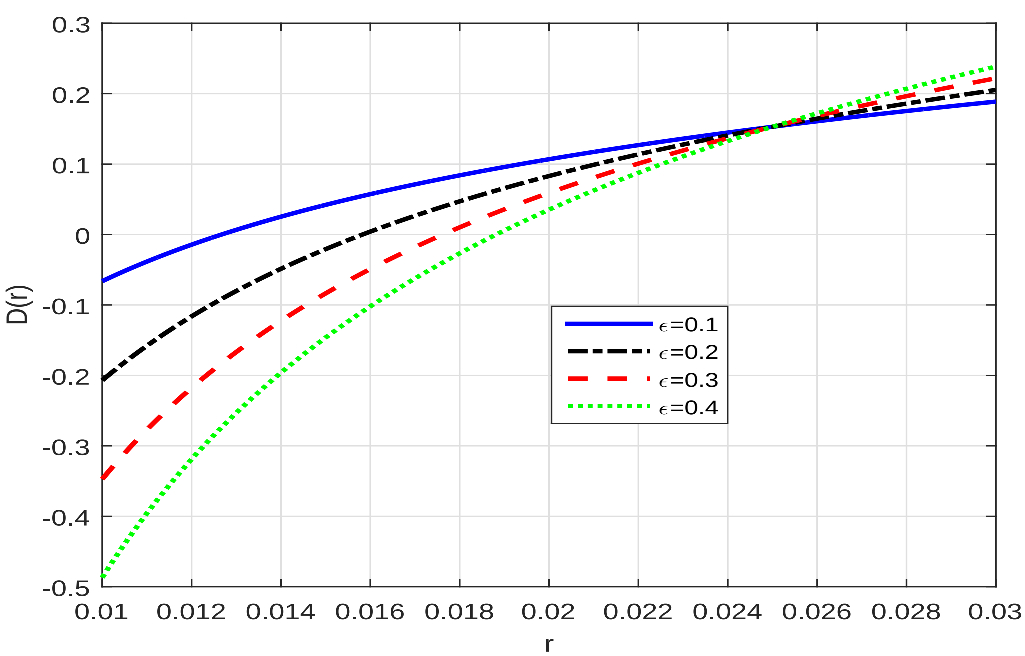

Figure 9: Effects of EIS on Aggregate Savings in GE

Table 1: Measures of the Risk Free Rate (1980-2010)

	Three-month Treasury Bills	One-year Treasury Bills
CPI (1981 – 2010)	1.87%	2.33%
PCE (1981 – 2010)	1.96%	2.42%
CPI (1981 – 2015)	1.37%	1.78%
PCE (1981 – 2015)	1.75%	2.16%

Table 2: Estimation and Calibration Results

	Parameter	Values
Discrete specification, eq. (29)		
constant	ϕ_0	0.0005
persistence	ϕ_1	0.919
std. of shock	σ	0.175
Continuous-time specification, eq. (1)		
persistence	ρ	0.082
std. of income changes	σ_y	0.182